On location of zeros in solution of second order complex differential equations

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The article is devoted to presenting new approaches to the study of the classical problem of finding the position of zero solutions of second order complex differential equations.

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1. Introduction and preliminaries

The revival of Rolf Nevannlina's idea from 1926 (see [1]), which appeared after the year 2000, brought us to the thought to use simplified Sturm's theorems on real field in giving a unique approach to an oscillating equation $y'' + A(x)\ y = 0$, $A(x) > 0$ and to, so called complex differential equation of oscillation

$$\frac{d^2w}{dz^2} + A(z)\, w = 0, \tag{1}$$

where $w(z)$ is function of one complex variable, as well as for the Vecua equation

$$\frac{\partial w}{\partial \overline{z}} = A(z, \overline{z})\, w + B(z, \overline{z})\, \overline{w}, \tag{2}$$

where $w = w(z, \overline{z})$ is function of two independent complex variables, by using partial equations, since (1) and (2) can be derived to appropriate systems of partial equations (for details see [2]). Zero solutions of the equations (1) and (2) would be found by reduction of these partial equations to ordinary differential equations by rays $y = kx$, where Sturm's theorems would be used.

The main difficulty is that equation (1) has solutions which can have only isolated zeros, while the equation (2) can have both isolated and non isolated zeros. However, the idea of reduction to ordinary differential equations has its adventages, not only in (1) but particularly in (2), where almost impassable obstacle is overcame, and this obstacle is the impossibility of quadrature solution in the case of conjugation of function $\overline{w}$. It is known that operation of conjugation of function $\overline{w}$ is not linear, because it also has rotation and not only translations.

2. The real partial equation for equation (1)

In order to develop this idea in the easiest way, for the equation of the second order (1), we will first take the simplest complex differential equation of the first order

$$\frac{dw}{dz} + A(z)\,w = 0, \tag{3}$$

where $A(z) = a(x,y) + ib(x,y)$ is given analytical function, while $w(z) = u(x,y) + iv(x,y)$ is unknown function, for which applies following

Theorem 1. *The equation (3) decomposes to the system of real partial equations*

$$\begin{aligned} \frac{\partial u}{\partial x} &= -a(x,y)\,u + b(x,y)\,v, \\ \frac{\partial v}{\partial x} &= -b(x,y)\,u - a(x,y)\,v. \end{aligned} \tag{4}$$

Theorem 2. *The system (4) derives to a partial equation of the second order*

$$\frac{\partial^2 \Pi}{\partial x^2} + \left(2a - \frac{\frac{\partial b}{\partial x}}{b}\right)\frac{\partial \Pi}{\partial x} + \left(b\left(\frac{a}{b}\right)' + a^2 + b^2\right)\Pi = 0, \tag{5}$$

where Π *is* $u(x,y)$ *or* $v(x,y)$.

Theorem 3. *Only by rays* $y = kx$, $k = \tan\varphi$, $0 \le \varphi \le \infty$ *in* xOy *— plane partial equations (5) is derived to ordinary differential equation*

$$\begin{aligned} &\frac{d^2 v}{dx^2} - \left(\frac{b' + ka'}{b + ka} - 2(a - bk)\right)\frac{dv}{dx} + \\ &+ \left(\frac{(b' + ka')(a - kb)}{b + ka} + (a - kb)^2 + (a' + kb')\right)v = 0, \end{aligned} \tag{6}$$

where Sturm's zero theorems can be applied.

By using this we could determine locations of zeros by each ray $y = kx$.

Now we shall go to the complex differential equation (1) from which we obtain system (7) which is similar to system (4), but now of second order

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= a(x,y)\,u - b(x,y)\,v, \\ \frac{\partial^2 v}{\partial x^2} &= b(x,y)\,u + a(x,y)\,v. \end{aligned} \tag{7}$$

This system can also be reduced to a single partial equation of the fourth order by elimination of v (or u) and double differentiation and then only by rays $y = kx$ can be reduced to ordinary differential equation of fourth order

$$\frac{d^4 u}{dx^4} = A^*(x)\frac{d^2 u}{dx^2} + B^*(x)\frac{du}{dx} + C^*(x)\,u(x,k). \tag{8}$$

This equation can have.

1. Four monotonous solutions, in which, by Sturm, zero solutions are either the only ones or they do not exist.

2. Two monotonous and two oscillating solutions. The monotonous can have maximum one zero, while the oscillating solutions have many zeros but they comply to Sturm's theorems.

3. All fourth solutions are oscillatory. Their separation by iterations is special work (for details see [3]).

Therefore, the procedures of solving the equation of the first order (3) and the second order (1) are similar, only the technical difficulties in solving the equation of the fourth order (8) are much greater.

3. The real partial equation for the Vecua equation (2)

The equation (2) can be reduced to system of partial equations

$$\begin{aligned} \frac{1}{2}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right) &= (a+c)\,u+(d-b)\,v, \\ \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) &= (b+d)\,u+(a-c)\,v. \end{aligned} \tag{9}$$

This system can also be reduced to a single partial equation of the second order, and then by rays $y = kx$ on real differential equations of second order, where Sturm's theorems on zero location are applied. By this, the main difficulty of the operation of conjugation $\overline{w}$ is reduced only to a sign in front of $c = c\,(x, y)$ or $d = d\,(x, y)$, which in real differential equations makes no difficulties.

Let us take the simplest inhomogeneous equation Vecua (2). It can be easily proved that the formal solution is

$$w\,(z,\overline{z}) = \exp\left(\hat{\int} A\,(z,\overline{z})\,d\overline{z}\right)\left[C\,(z) + \hat{\int} F\,(z,\overline{z})\exp\left(-\hat{\int} A\,(z,\overline{z})\,d\overline{z}\right)d\overline{z}\right], \tag{10}$$

where $\hat{\int}$ means that it is integrated only by conjugated complex variable, while $C\,(z)$ is arbitrary analytical function, in the role of generalized integrating "constant".

From the formal solution (10) we are still far from zero solutions, both isolated and non isolated. It is interesting that this question has not been initiated in literature so far! That is why, due to seriousness of the problem, for now we shall give only a few basic examples.

Example 1. The Vecua equation $\dfrac{\partial w}{\partial \overline{z}} = 1$, $(A\,(z,\overline{z}) = 0,\ B\,(z,\overline{z}) = 1)$ has a formal solution, based on (10),

$$w\,(z,\overline{z}) = \overline{z} + C\,(z)\,,$$

where $C\,(z)$ is an analytical function. For each choice $C\,(z)$ the issue of zeros changes significantly. If we take $C\,(z) = -e^{z}$, then the particular solution $w(z,\overline{z})$ has zeros for $u\,(x, y) = x - e^{x}\cos y = 0$, $v\,(x, y) = -y - e^{x}\sin y = 0$. So, by eliminating y, we get one transcendent equation for locations of abscissa of zeros $\dfrac{x}{e^{x}} = \cos\sqrt{e^{2x} - x^{2}}$. Graphically, by using the curves $y_1 = e^{2x} - x^2$, $y_2 = \sqrt{y_1}$, $y_3 = \frac{x}{e^x}$, $y_4 = \cos\sqrt{y_1}$ we can calculate typical Sturm's zeros.

Example 2. For the same Vecua equation we used in previous example, $\frac{\partial w}{\partial \overline{z}} = 1$, for $C(z) = -z^2$, we will have an equation for zeros $\overline{z} - z^2 = 0$. From here we obtain four isolated zeros: $M_1(0,0)$, $M_2(1,0)$, $M_3\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $M_4\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

References

[1] Laine I. Nevanlinna Theory and Complex Differential Equations. Berlin: Walter de Guyter, 1997.

[2] 200 Godina Kvalitativne Analize Diferencijalnih Jednačina. Teoreme Šturma / D. Dimitrovski, M. Rajović, S. Cvejić, M. Lekić, V. Rajović, A. Dimitrovski. Kosovska Mitrovica, 2008.

[3] Mitevska J., Dimitrovski D. Linearni Trigonometrii od Cetvrti i Sesti Red. Posebni Izdanija. Knjiga II (23). Skopje, 1996.

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