COLLOCATION METHODS FOR SYSTEMS OF CAUCHY SINGULAR INTEGRAL EQUATIONS ON AN INTERVAL

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Dedicated to Erhard Meister on the occasion of his seventieth birthday

Получены необходимые и достаточные условия устойчивости методов коллокации по узлам Чебышева первого и второго рода для скалярных сингулярных интегральных уравнений Коши с кусочно-непрерывными коэффициентами на отрезке и для систем таких уравнений. Рассматривается также поведение сингулярных значений матриц дискретных уравнений.

1. Introduction

Recently a collocation method, which is based on the Chebyshev nodes of second kind as collocation points and on approximating the solution by polynomials multiplied with the Chebyshev weight of second kind, was studied for both linear and nonlinear Cauchy singular integral equations (CSIE's) on the interval $[-1, 1]$ (see [11, 12, 22] for the linear case and [9] for the nonlinear case). There are several reasons for choosing Chebyshev nodes as collocation points independently from the asymptotic of the solution of the CSIE. At first we get a very cheap preprocessing for the construction of the matrix of the discretized equation, which is especially important in case of approximating the solution of a nonlinear CSIE by a sequence of solutions of linear equations (cf. [9]). A second reason is the possibility to apply such collocation methods to systems of CSIE's, which is, in some sense, the main topic of the present paper. Indeed, in [11] there are only given necessary and sufficient conditions for the stability of the mentioned collocation method in the case of scalar CSIE's of the form

$$a(x)u(x) + \frac{b(x)}{\pi i}\int_{-1}^{1}\frac{u(y)}{y-x}\,dy = f(x)\,, \quad -1 < x < 1\,, \tag{1.1}$$

where a and b are given piecewise continuous functions and the equation is considered in an appropriate weighted $\mathbf{L}^2$-space $\mathbf{L}^2_\sigma$. In [12, 22] the system case could only be investigated under additional conditions on the coefficients of the singular integral operator. In the present paper we study a more general situation, namely we give necessary and sufficient conditions for the stability of operator sequences $\{A_n\}$ belonging to a C^*-algebra $\mathcal{A}$, which is generated by the sequences of the collocation method for equations of type (1.1). These stability conditions can be formulated in the following way. There exist *-homomorphisms $W : \mathcal{A} \longrightarrow \mathcal{L}(\mathbf{L}^2_\sigma)$, $\widetilde{W} : \mathcal{A} \longrightarrow \mathcal{L}(\mathbf{L}^2_\sigma)$ and $\eta_\pm : \mathcal{A} \longrightarrow \mathcal{L}(\ell^2)$ such that, in case of collocation w.r.t. Chebyshev nodes of second kind, a sequence $\{A_n\} \in \mathcal{A}$ is stable if and only if the operators $W\{A_n\}$, $\widetilde{W}\{A_n\}$, and $\eta_\pm\{A_n\}$ are invertible. In case of collocation w.r.t. Chebyshev nodes of first kind the invertibility of $W\{A_n\}$ and $\widetilde{W}\{A_n\}$ is necessary and sufficient for the stability of $\{A_n\} \in \mathcal{A}$. It is important that such stability results for sequences belonging to an algebra $\mathcal{A}$ can be extended to the case of systems of CSIE's.

The paper is organized as follows. In Section 2 the collocation method is described, where we also consider Chebyshev nodes of first kind as collocation points. In Section 3 some basic facts are collected and the existence of several strong limits of the involved operator sequences is established. The main result is proved in Section 4 using localization principles in C^*-algebras. In the opinion of the authors it seems to be surprising that the stability conditions in the two cases of Chebyshev nodes of first and second kind are very different. The results of Section 4 are used in Section 5 to describe the behaviour of the smallest singular values of the operator sequences of the collocation method w.r.t. the Chebyshev nodes of second kind. The last Section 6 is dedicated to the very technical proof of a lemma on the local spectrum of the sequence of the collocation method in case of the Chebyshev nodes of first kind.

2. A polynomial collocation method

Let $\sigma(x) = (1-x^2)^{-1/2}$ and $\varphi(x) = (1-x^2)^{1/2}$ denote the Chebyshev weights of first and second kind on the interval $(-1, 1)$, respectively, and let $\mathbf{L}^2_\sigma$ refer to the Hilbert space of all w.r.t. σ on $(-1,1)$ square integrable functions, equipped with inner product and norm

$$\langle u, v\rangle_\sigma = \int_{-1}^1 u(x)\overline{v(x)}\sigma(x)\,dx \quad \text{and} \quad \|u\|_\sigma := \sqrt{\langle u, u\rangle_\sigma}.$$

For $\omega \in \{\sigma, \varphi\}$ and $n \ge 0$, let p_n^ω stand for the w.r.t. ω orthonormal polynomial of degree n and abbreviate p_n^σ and p_n^φ to T_n and U_n, respectively. It is well known that

$$T_0(x) = \frac{1}{\sqrt{\pi}}, \quad T_n(\cos s) = \sqrt{2/\pi}\cos ns, \quad n \ge 1,\ s \in (0,\pi),$$

and

$$U_n(\cos s) = \sqrt{2/\pi}\frac{\sin(n+1)s}{\sin s}, \quad n \ge 0,\ s \in (0,\pi).$$

Further define weighted polynomials $\widetilde{u}_n := \varphi U_n$. Both $\{T_n\}_{n=0}^\infty$ and $\{\widetilde{u}_n\}_{n=0}^\infty$ form an orthonormal basis in $\mathbf{L}^2_\sigma$. The zeros of p_n^ω are known to be

$$x_{jn}^\sigma = \cos\frac{2j-1}{2n}\pi \quad \text{and} \quad x_{jn}^\varphi = \cos\frac{j\pi}{n+1} \quad \text{where} \quad j = 1, \ldots, n.$$

Further, the Lagrange interpolation operator L_n^ω acts on a function $f : (-1,1) \longrightarrow \mathbb{C}$ by

$$L_n^\omega f = \sum_{j=1}^n f(x_{jn}^\omega)\ell_{jn}^\omega, \quad \ell_{jn}^\omega(x) = \prod_{k=1,k\ne j}^n \frac{x - x_{kn}^\omega}{x_{jn}^\omega - x_{kn}^\omega} = \frac{p_n^\omega(x)}{(x - x_{jn}^\omega)(p_n^\omega)'(x_{jn}^\omega)}.$$

A function $a : [-1,1] \longrightarrow \mathbb{C}$ is called piecewise continuous if it is continuous at ± 1 and if the one-sided limits $a(x \pm 0)$ exist and satisfy $a(x-0) = a(x)$ for all $x \in (-1,1)$. The set of all piecewise continuous functions on $[-1,1]$ is denoted by $\mathbf{PC} = \mathbf{PC}[-1,1]$.

For given functions $a, b \in \mathbf{PC}$ and $f \in \mathbf{L}^2_\sigma$, consider the Cauchy singular integral equation

$$a(x)u(x) + \frac{b(x)}{\pi i} \int_{-1}^{1} \frac{u(y)}{y-x}\, dy = f(x)\,, \quad -1 < x < 1\,. \tag{2.1}$$

Both the Cauchy singular integral operator

$$S : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma\,, \quad u \mapsto \frac{1}{\pi i} \int_{-1}^{1} \frac{u(y)}{y - \cdot}\, dy$$

and the multiplication operators $aI : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$, $u \mapsto au$, belong to the algebra $\mathcal{L}(\mathbf{L}^2_\sigma)$ of all linear and bounded operators in $\mathbf{L}^2_\sigma$ which justifies to consider equation (2.1) on this space. For the approximate solution of (2.1), we look for a function $u \in \mathbf{L}^2_\sigma$ of the form

$$u_n = \sum_{k=0}^{n-1} \xi_{kn} \widetilde{u}_k\,, \quad \xi_n = [\xi_{kn}]_{k=0}^{n-1} \in \mathbb{C}^n$$

which satisfies the collocation system

$$a(x^\omega_{jn})u_n(x^\omega_{jn}) + \frac{b(x^\omega_{jn})}{\pi i} \int_{-1}^{1} \frac{u(y)}{y - x^\omega_{jn}}\, dy = f(x^\omega_{jn})\,, \quad j = 1, \ldots, n\,. \tag{2.2}$$

If we introduce Fourier projections

$$P_n : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma\,, \quad u \mapsto \sum_{k=0}^{n-1} \langle u, \widetilde{u}_k \rangle_\sigma \widetilde{u}_k$$

and weighted interpolation operators $M^\omega_n := \varphi L^\omega_n \varphi^{-1}$, then the collocation system (2.2) can be rewritten as an operator equation

$$M^\omega_n (aI + bS) P_n u_n = M^\omega_n f\,, \quad u_n \in \operatorname{im} P_n\,. \tag{2.3}$$

The reason for using M^ω_n instead of L^ω_n is that the range of M^ω_n coincides with that one of the Fourier projection P_n.

Our main concern is the stability of the sequence $\{A_n\}$ with $A_n = M^\omega_n A P_n$ and $A = aI + bS$. Recall that a sequence $\{A_n\}$ is stable if there exists an n_0 such that the operators $A_n : \operatorname{im} P_n \longrightarrow \operatorname{im} P_n$ are invertible for $n \geq n_0$ and that their inverses A_n^{-1} are uniformly bounded:

$$\sup_{n \geq n_0} \left\| A_n^{-1} P_n \right\|_{\mathbf{L}^2_\sigma \to \mathbf{L}^2_\sigma} < \infty\,.$$

If the sequence $\{A_n\}$ is stable and if $u^* \in \mathbf{L}^2_\sigma$ and $u^*_n \in \operatorname{im} P_n$ are the solutions of (2.1) and (2.3), respectively, then the estimate

$$\|P_n u^* - u^*_n\|_\sigma \leq \left\| A_n^{-1} P_n \right\|_{\mathbf{L}^2_\sigma \to \mathbf{L}^2_\sigma} \Big(\|A_n P_n u^* - A u^*\|_\sigma + \|f - M^\omega_n f\|_\sigma \Big)$$

shows that u^*_n converges to u^* in the norm of $\mathbf{L}^2_\sigma$ if the method (2.3) is consistent, i. e. if $A_n P_n \longrightarrow A$ (strong convergence) and if $M^\omega_n f \longrightarrow f$ (convergence in $\mathbf{L}^2_\sigma$). The stability result for the collocation method (2.3), which is a conclusion of Theorem 4.8, reads as follows.

Theorem 2.1. *Let $a, b \in \mathbf{PC}$. Then the sequence $\{M_n^\sigma(aI+bS)P_n\}$ is stable if and only if the operator $aI+bS$ is invertible on $\mathbf{L}^2_\sigma$, and the sequence $\{M_n^\varphi(aI+bS)P_n\}$ is stable if and only if the operators $aI+bS$ and $aI-bS$ are invertible on $\mathbf{L}^2_\sigma$.*

Our main tools for studying the stability of an approximation sequence are the translation of the stability problem into an invertibility problem in a suitable C^*-algebra and the application of local principles (see, for example, [7, Chapter 3] and [16, Chapter 7]).

For the algebraization of the stability problem, let $\mathcal{F}$ denote the C^*-algebra of all bounded sequences $\{A_n\}$ of linear operators $A_n : \operatorname{im} P_n \longrightarrow \operatorname{im} P_n$, provided with the supremum norm $\|\{A_n\}\|_{\mathcal{F}} := \sup_{n\ge 1}\|A_n\|_{\mathbf{L}^2_\sigma\to\mathbf{L}^2_\sigma}$ and with operations $\{A_n\}+\{B_n\} := \{A_n+B_n\}$, $\{A_n\}\{B_n\} := \{A_nB_n\}$, and $\{A_n\}^* := \{A_n^*\}$. Further, let $\mathcal{N}$ be the two-sided closed ideal of $\mathcal{F}$ consisting of all sequences $\{C_n\}\in\mathcal{F}$ such that $\lim_{n\to\infty}\|C_nP_n\|_{\mathbf{L}^2_\sigma\to\mathbf{L}^2_\sigma} = 0$. Then a simple Neumann series argument shows that the sequence $\{A_n\}\in\mathcal{F}$ is stable if and only if the coset $\{A_n\}+\mathcal{N}$ is invertible in the quotient algebra $\mathcal{F}/\mathcal{N}$. In the case at hand, it is more convenient to work in a subalgebra of $\mathcal{F}$ rather than in $\mathcal{F}$ itself (the main point being that the ideal $\mathcal{N}$ proves to be too small for the sake of localization). To introduce this subalgebra, define operators

$$W_n : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma, \quad u \mapsto \sum_{j=0}^{n-1}\langle u, \widetilde{u}_{n-1-j}\rangle_\sigma \widetilde{u}_j,$$

and consider the set $\mathcal{F}^W$ of all sequences $\{A_n\}\in\mathcal{F}$, for which the strong limits

$$W\{A_n\} := \text{s-lim}\, A_nP_n\,, \quad (W\{A_n\})^* = \text{s-lim}\, A_n^*P_n\,, \tag{2.4}$$

and

$$\widetilde{W}\{A_n\} := \text{s-lim}\, W_nA_nW_n\,, \quad (\widetilde{W}\{A_n\})^* = \text{s-lim}\,(W_nA_nW_n)^*P_n \tag{2.5}$$

exist. Furthermore, let $\mathcal{J}$ refer to the collection of all sequences $\{A_n\}$ of the form

$$A_n = P_nK_1P_n + W_nK_2W_n + C_n \quad \text{with} \quad K_j\in\mathcal{K}(\mathbf{L}^2_\sigma)\,,\ \{C_n\}\in\mathcal{N}\,,$$

where $\mathcal{K}(\mathbf{L}^2_\sigma)\subset\mathcal{L}(\mathbf{L}^2_\sigma)$ stands for the ideal of all compact operators.

Lemma 2.2. (*a*) *$\mathcal{F}^W$ is a C^*-subalgebra of $\mathcal{F}$, and $\mathcal{J}$ is a closed two-sided ideal of $\mathcal{F}^W$.*

(*b*) *A sequence $\{A_n\}\in\mathcal{F}^W$ is stable if and only if the operators $W\{A_n\}, \widetilde{W}\{A_n\} : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$ and the coset $\{A_n\}+\mathcal{J}\in\mathcal{F}^W/\mathcal{J}$ are invertible.*

The proof is not hard and can be found in [21], Prop. 3, or in [16], Theorem 7.7. It rests essentially on the weak convergence of the sequence $\{W_n\}$ to zero.

3. Consistency of the method

The goal of this section is to show that the method (2.3) is consistent with equation (2.1) in the sense that $M_n^\omega f \longrightarrow f$ in $\mathbf{L}^2_\sigma$ under suitable restrictions for f and that $A_nP_n \longrightarrow A$ strongly. The proof of the approximation properties of the interpolation operators M_n^ω is based on the following auxiliary results.

Lemma 3.1 ([14], Theor. 9.25).. *Let μ,ν be classical Jacobi weights with $\mu\nu \in \mathbf{L}^1(-1,1)$ and let $j \in \mathbb{N}$ be fixed. Then, for each polynomial q with $\deg q \le jn$,*

$$\sum_{k=1}^{n} \lambda_{kn}^{\mu} |q(x_{kn}^{\mu})| \nu(x_{kn}^{\mu}) \le \mathrm{const} \int_{-1}^{1} |q(x)|\mu(x)\nu(x)\,dx\,,$$

where the constant does not depend on n and q and where x_{kn}^{μ} and $\lambda_{kn}^{\mu} = \int_{-1}^{1} \ell_{kn}^{\mu}(x)\mu(x)\,dx$ are the nodes and the Christoffel numbers of the Gaussian rule w.r.t. the weight μ, respectively.

Let Q_n^{μ} denote the Gaussian quadrature rule w.r.t. the weight μ,

$$Q_n^{\mu} f = \sum_{k=1}^{n} \lambda_{kn}^{\mu} f(x_{kn}^{\mu}),$$

and write $\mathbf{R} = \mathbf{R}(-1,1)$ for the set of all functions $f : (-1,1) \longrightarrow \mathbb{C}$, which are bounded and Riemann integrable on each interval $[\alpha,\beta] \subset (-1,1)$.

Lemma 3.2 ([5], Satz III.1.6b and Satz III.2.1).. *Let $\mu(x) = (1-x)^{\gamma}(1+x)^{\delta}$ with $\gamma,\delta > -1$. If $f \in \mathbf{R}$ satisfies*

$$|f(x)| \le \mathrm{const}\,(1-x)^{\varepsilon-1-\gamma}(1+x)^{\varepsilon-1-\delta}\,, \quad -1 < x < 1\,,$$

for some $\varepsilon > 0$, then $\lim\limits_{n\to\infty} Q_n^{\mu} f = \int_{-1}^{1} f(x)\mu(x)\,dx$. If even

$$|f(x)| \le \mathrm{const}\,(1-x)^{\varepsilon-\frac{1+\gamma}{2}}(1+x)^{\varepsilon-\frac{1+\delta}{2}}\,, \quad -1 < x < 1\,,$$

then $\lim_{n\to\infty} \|f - L_n^{\mu} f\|_{\mu} = 0$.

Corollary 3.3. *Let $f \in \mathbf{R}$ and $|f(x)| \le \mathrm{const}\,(1-x^2)^{\varepsilon-\frac{1}{4}}$, $-1 < x < 1$, for some $\varepsilon > 0$. Then $M_n^{\omega} f \longrightarrow f$ in $\mathbf{L}^2_{\sigma}$ for $\omega = \varphi$ and $\omega = \sigma$.*

Proof. Since $\|f - M_n^{\omega} f\|_{\sigma} = \|\varphi^{-1} f - L_n^{\omega}\varphi^{-1} f\|_{\varphi}$, we can immediately apply the second assertion of Lemma 3.2 to get the assertion in case $\omega = \varphi$. To consider the case $\omega = \sigma$, introduce the quadrature rule

$$Q_n f = \int_{-1}^{1} (L_n^{\sigma} f)(x)\varphi(x)\,dx = \sum_{k=1}^{n} \sigma_{kn} f(x_{kn}^{\sigma})\,,$$

where

$$\sigma_{kn} = \int_{-1}^{1} \frac{T_n(x)}{x - x_{kn}^{\sigma}} \frac{\varphi(x)}{T_n'(x_{kn}^{\sigma})}\,dx = \int_{-1}^{1} \frac{T_n(x)(1-x^2)\sigma(x)}{(x - x_{kn}^{\sigma})T_n'(x_{kn}^{\sigma})}\,dx = \frac{\pi\,[1 - (x_{kn}^{\sigma})^2]}{n}$$

for $n > 2$. Consequently,

$$Q_n f = \frac{\pi}{n} \sum_{k=1}^{n} \left[1 - (x_{kn}^{\sigma})^2\right] f(x_{kn}^{\sigma})\,.$$

Since the nodes x_{kn}^{σ} of the quadrature rule Q_n are the zeros of $2T_n = U_n - U_{n-2}$, the estimate

$$\int_{-1}^{1} |(L_n^{\sigma} f)(x)|^2\,\varphi(x)\,dx \le 2\,Q_n |f|^2$$

holds true (see [5, Hilfssatz 2.4, § III.2]). As an immediate consequence we obtain

$$\|M_n^\sigma f\|_\sigma^2 = \left\|L_n^\sigma \varphi^{-1} f\right\|_\varphi^2 \le \frac{2\pi}{n}\sum_{k=1}^n |f(x_{kn}^\sigma)|^2 = 2\,Q_n^\sigma |f|^2 . \tag{3.1}$$

Now let $\delta > 0$ be arbitrary and p be a polynomial such that $\|\varphi p - f\|_\sigma < \delta$. For $n > \deg p$ we have $\|M_n^\sigma f - f\|_\sigma^2 \le 2\left(\|M_n^\sigma(\varphi p - f)\|_\sigma^2 + \|\varphi p - f\|_\sigma^2\right)$. Since, in view of Lemma 3.2, $\lim_{n\to\infty} Q_n^\sigma |\varphi p - f|^2 = \|\varphi p - f\|_\sigma^2$, we get via (3.1) that $\limsup\limits_{n\to\infty} \|M_n^\sigma f - f\|_\sigma^2 < 6\,\delta^2$, which proves the assertion in second case, too.

The strong convergence of the sequence $\{M_n^\omega(aI + bS)P_n\}$ is part of the assertion of the following theorem. For the description of the occuring strong limits we need two further operators: the isometry

$$J_\sigma : \mathbf{L}_\sigma^2 \longrightarrow \mathbf{L}_\sigma^2 , \quad u \longrightarrow \sum_{n=0}^\infty \gamma_n \langle u, \widetilde{u}_n\rangle_\sigma T_n , \tag{3.2}$$

where $\gamma_0 = \sqrt{2}$ and $\gamma_n = 1$ for $n \ge 1$, and the shift operator

$$V : \mathbf{L}_\sigma^2 \longrightarrow \mathbf{L}_\sigma^2 , \quad u \mapsto \sum_{n=0}^\infty \langle u, \widetilde{u}_n\rangle_\sigma \widetilde{u}_{n+1} \tag{3.3}$$

with its adjoint $V^* : \mathbf{L}_\sigma^2 \longrightarrow \mathbf{L}_\sigma^2$, $u \mapsto \sum_{n=0}^\infty \langle u, \widetilde{u}_{n+1}\rangle_\sigma \widetilde{u}_n$.

Theorem 3.4. *For $a, b \in \mathbf{PC}$, the sequence $\{A_n^\omega\} := \{M_n^\omega(aI + bS)P_n\}$ belongs to the algebra $\mathcal{F}^W$. In particular, $W\{A_n^\omega\} = aI + bS$ and*

$$\widetilde{W}\{A_n^\omega\} = \begin{cases} aI - bS, & \omega = \varphi , \\ J_\sigma^{-1}(aJ_\sigma + ibV^*), & \omega = \sigma . \end{cases}$$

Proof. *Step 1: Uniform boundedness.* Let $a, b \in \mathbf{PC}$ and $\omega \in \{\varphi, \sigma\}$. We have to verify the strong convergence of each of the sequences $\{M_n^\omega(aI + bS)P_n\}$, $\{(M_n^\omega(aI + bS)P_n)^*\}$, $\{W_n M_n^\omega(aI + bS)W_n\}$ and $\{(W_n M_n^\omega(aI + bS)W_n)^*\}$. Let us start with showing the uniform boundedness of these sequences. Since $\|W_n\| = 1$, it is sufficient to prove the uniform boundedness of the sequences $\{M_n^\omega(aI + bS)P_n\}$.

We write $M_n^\omega bSP_n$ as $M_n^\omega bP_n M_n^\omega SP_n$ and consider first sequences of the form $M_n^\omega aP_n$ where a is an arbitrary function with $\|a\|_\infty := \sup\{|a(x)| : x \in [-1, 1]\} < \infty$. Let $u_n = \varphi v_n \in \operatorname{im} P_n$. Then, using the algebraic accuracy of a Gaussian rule, we get the estimate

$$\|M_n^\varphi a u_n\|_\sigma^2 = \|L_n^\varphi a v_n\|_\varphi^2 = Q_n^\varphi |a v_n|^2 \le \|a\|_\infty^2 \|v_n\|_\varphi^2 = \|a\|_\infty^2 \|u_n\|_\sigma^2 . \tag{3.4}$$

In case $\omega = \sigma$ we apply Relation (3.1) and Lemma 3.1 with $\mu = \sigma$ and $\nu(x) = 1 - x^2$ to obtain

$$\|M_n^\sigma a u_n\|_\sigma^2 \le 2\,\|a\|_\infty \frac{\pi}{n}\sum_{k=1}^n \left[1 - (x_{kn}^\sigma)^2\right] |v_n(x_{kn}^\sigma)|^2 \le \text{const}\ \|a\|_\infty^2 \|v_n\|_\varphi^2$$

and thus

$$\|M_n^\sigma a u_n\|_\sigma \le \text{const}\ \|a\|_\infty \|u_n\|_\sigma , \quad u_n \in \operatorname{im} P_n , \tag{3.5}$$

where the constant does not depend on a, n, and u_n.

For the uniform boundedness of the sequences $\{M_n^\omega S P_n\}$ we observe that

$$S\varphi U_n = iT_{n+1}\,, \quad n = 0, 1, 2, \ldots, \tag{3.6}$$

which shows that, for $u_n \in \operatorname{im} P_n$, the function $q_n := S u_n$ is a polynomial of degree not greater than n. Thus, Relation (3.1), Lemma 3.1, and the boundedness of $S : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$ yield

$$\|M_n^\sigma S u_n\|_\sigma^2 \le 2Q_n^\sigma |q_n|^2 \le \text{const}\ \|q_n\|_\sigma^2 \le \text{const}\ \|u_n\|_\sigma^2\,.$$

In case $\omega = \varphi$ we also use Lemma 3.1 to obtain

$$\|M_n^\varphi S u_n\|_\sigma^2 = \left\|L_n^\varphi \varphi^{-1} q_n\right\|_\varphi^2 = \sum_{k=1}^n \lambda_{kn}^\varphi |q_n(x_{kn}^\varphi)|^2 \left[1 - (x_{kn}^\varphi)^2\right]^{-1} \le \text{const}\ \|q_n\|_\sigma^2\,.$$

This verifies the uniform boundedness of all sequences under consideration. Their strong convergence on $\mathbf{L}^2_\sigma$ follows once we have shown their convergence on all basis functions $\widetilde{u}_n$ of $\mathbf{L}^2_\sigma$.

Step 2: Convergence of $\{M_n^\omega(aI + bS)P_n\}$. It is an immediate consequence of Corollary 3.3 and of (3.6) that

$$\lim_{n\to\infty} M_n^\omega(aI + bS)P_n\widetilde{u}_m = (aI + bS)\widetilde{u}_m \quad \text{in} \quad \mathbf{L}^2_\sigma \quad \text{for all} \quad m = 0, 1, 2, \ldots$$

Step 3: Convergence of $\{(M_n^\omega(aI + bS)P_n)^*\}$. The determination of the adjoint sequence is based upon a formula for the Fourier coefficients of the interpolating function $M_n^\omega f$. For this goal, we write

$$M_n^\omega f = \sum_{j=0}^{n-1} \alpha_{jn}^\omega(f)\widetilde{u}_j$$

and get in case of $\omega = \varphi$

$$\alpha_{jn}^\varphi(f) = \langle M_n^\varphi f, \widetilde{u}_j\rangle_\sigma = \langle L_n^\varphi \varphi^{-1} f, U_j\rangle_\varphi = \frac{\pi}{n+1}\sum_{k=1}^n f(x_{kn}^\varphi)\widetilde{u}_j(x_{kn}^\varphi)\,, \tag{3.7}$$

$j = 0, \ldots, n-1$. In case of $\omega = \sigma$ we have for $j = 0, \ldots, n-2$

$$\alpha_{jn}^\sigma(f) = \langle M_n^\sigma f, \widetilde{u}_j\rangle_\sigma = \langle L_n^\sigma \varphi^{-1} f, \varphi^2 U_j\rangle_\sigma = \frac{\pi}{n}\sum_{k=1}^n f(x_{kn}^\sigma)\widetilde{u}_j(x_{kn}^\sigma)\,.$$

For $j = n-1$ we use the three-term recurrence relation

$$U_{k+1}(x) = 2xU_k(x) - U_{k-1}(x)\,, \quad k = 1, 2, \ldots, \tag{3.8}$$

as well as the relation

$$T_{n+1}(x) = \frac{1}{2}[U_{n+1}(x) - U_{n-1}(x)]\,, \quad n = 0, 1, 2, \ldots,\ U_{-1} \equiv 0\,, \tag{3.9}$$

to obtain

$$(1 - x^2)U_{n-1}(x) = \frac{1}{2}\left[T_{n-1}(x) - T_{n+1}(x)\right]\,. \tag{3.10}$$

Consequently, with $\psi(x) := x$,

$$\begin{aligned} \alpha^\sigma_{n-1,n}(f) &= \langle L^\sigma_n \varphi^{-1} f, \varphi^2 U_{n-1}\rangle_\sigma = \frac{1}{2}\langle L^\sigma_n \varphi^{-1} f, T_{n-1}\rangle_\sigma = \\ &= \frac{\pi}{2n}\sum_{k=1}^n \frac{f(x^\sigma_{kn})}{\varphi(x^\sigma_{kn})} T_{n-1}(x^\sigma_{kn}) = \frac{\pi}{2n}\sum_{k=1}^n f(x^\sigma_{kn})\widetilde{u}_{n-1}(x^\sigma_{kn}) . \end{aligned}$$

Thus,

$$\alpha^\sigma_{jn}(f) = \varepsilon_{jn}\frac{\pi}{n}\sum_{k=1}^n f(x^\sigma_{kn})\widetilde{u}_j(x^\sigma_{kn}) , \quad j = 0, \ldots, n-1 , \tag{3.11}$$

where $\varepsilon_{jn} = 1$ for $j = 0, \ldots, n-2$ and $\varepsilon_{jn} = 1/2$ for $j = n-1$. As an immediate consequence of (3.7) we deduce for $u, v \in \mathbf{L}^2_\sigma$

$$\begin{aligned} \langle M^\varphi_n a P_n u, v\rangle_\sigma &= \sum_{j=0}^{n-1} \frac{\pi}{n+1}\sum_{k=1}^n a(x^\varphi_{kn}) \sum_{\ell=0}^{n-1} \langle u, \widetilde{u}_\ell\rangle_\sigma \widetilde{u}_\ell(x^\varphi_{kn})\widetilde{u}_j(x^\varphi_{kn})\overline{\langle v, \widetilde{u}_j\rangle_\sigma} = \\ &= \overline{\sum_{\ell=0}^{n-1} \frac{\pi}{n+1}\sum_{k=1}^n \overline{a(x^\varphi_{kn})} \sum_{j=0}^{n-1} \langle v, \widetilde{u}_j\rangle_\sigma \widetilde{u}_j(x^\varphi_{kn})\widetilde{u}_\ell(x^\varphi_{kn})\langle u, \widetilde{u}_\ell\rangle_\sigma} = \\ &= \langle u, M^\varphi_n \overline{a} P_n v\rangle_\sigma . \end{aligned}$$

Hence, the adjoint of $M^\varphi_n a P_n : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$ is $M^\varphi_n \overline{a} P_n : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$. In case $\omega = \sigma$, (3.11) implies

$$\begin{aligned} \langle M^\sigma_n a P_n u, v\rangle_\sigma &= \sum_{j=0}^{n-1} \varepsilon_{jn}\frac{\pi}{n}\sum_{k=1}^n a(x^\sigma_{kn}) \sum_{\ell=0}^{n-1} \langle u, \widetilde{u}_\ell\rangle_\sigma \widetilde{u}_\ell(x^\sigma_{kn})\widetilde{u}_j(x^\sigma_{kn})\overline{\langle v, \widetilde{u}_j\rangle_\sigma} = \\ &= \overline{\sum_{\ell=0}^{n-1} \frac{\pi}{n}\sum_{k=1}^n \overline{a(x^\sigma_{kn})} \sum_{j=0}^{n-1} \varepsilon_{jn}\langle v, \widetilde{u}_j\rangle_\sigma \widetilde{u}_j(x^\sigma_{kn})\widetilde{u}_\ell(x^\sigma_{kn})\langle u, \widetilde{u}_\ell\rangle_\sigma} = \\ &= \langle u, (2P_n - P_{n-1}) M^\varphi_n \overline{a}\frac{1}{2}(P_{n-1} + P_n) v\rangle_\sigma . \end{aligned}$$

Thus,

$$(M^\varphi_n a P_n)^* = M^\varphi_n \overline{a} P_n \quad \text{and} \quad (M^\sigma_n a P_n)^* = \left(P_n - \frac{1}{2}P_{n-1}\right) M^\sigma_n \overline{a}(P_{n-1} + P_n) , \tag{3.12}$$

whence in both cases the strong convergence on $\mathbf{L}^2_\sigma$ of $(M^\omega_n a P_n)^*$ to $\overline{a}I$. For the determination of the adjoint operator of $M^\omega_n S P_n$, we recall the Poincaré-Bertrand commutation formula (see [13, Chapter II, Theorem 4.4]): If $\rho(x) = (1-x)^\alpha(1+x)^\beta$ is a Jacobi weight with $\alpha, \beta \in (-1, 1)$ then, for $u \in \mathbf{L}^2_\rho$ and $v \in \mathbf{L}^2_{\rho^{-1}}$,

$$\langle Su, v\rangle = \langle u, Sv\rangle , \tag{3.13}$$

where $\langle ., .\rangle$ refers to the unweighted $\mathbf{L}^2(-1, 1)$ inner product. Thus, the adjoint operator of $S : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$ is $\varphi S \varphi^{-1} I : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$. Taking into account that $SP_n u$ is a polynomial of degree at most n due to (3.6), we conclude that, for all $u, v \in \mathbf{L}^2_\sigma$,

$$\begin{aligned}\langle M_n^\varphi SP_n u, v\rangle_\sigma &= \langle L_n^\varphi \varphi^{-1} SP_n u, \varphi^{-1} P_n v\rangle_\varphi = \frac{\pi}{n+1}\sum_{k=1}^{n}(SP_n u)(x_{kn}^\varphi)(P_n v)(x_{kn}^\varphi) = \\ &= \langle SP_n u, L_n^\varphi \varphi^{-2} P_n v\rangle_\varphi = \langle SP_n u, \varphi M_n^\varphi \varphi^{-1} P_n v\rangle_\sigma = \\ &= \langle u, P_n \varphi S M_n^\varphi \varphi^{-1} P_n v\rangle_\sigma .\end{aligned}$$

Analogously we get, for $j = 0, \ldots, n-2$ and $u \in \mathbf{L}^2_\sigma$,

$$\begin{aligned}\langle M_n^\sigma SP_n u, \widetilde{u}_j\rangle_\sigma &= \langle L_n^\sigma \varphi^{-1} SP_n u, \varphi^2 U_j\rangle_\sigma = \frac{\pi}{n}\sum_{k=1}^{n}(SP_n u)(x_{kn}^\sigma)\widetilde{u}_j(x_{kn}^\sigma) = \langle SP_n u, L_n^\sigma \widetilde{u}_j\rangle_\sigma = \\ &= \langle u, P_n \varphi S \varphi^{-1} L_n^\sigma \widetilde{u}_j\rangle_\sigma\end{aligned}$$

and, again using Relation (3.10),

$$\begin{aligned}\langle M_n^\sigma SP_n u, \widetilde{u}_{n-1}\rangle_\sigma &= \frac{1}{2}\langle L_n^\sigma \varphi^{-1} SP_n u, \psi U_{n-2} - U_{n-3}\rangle_\sigma = \frac{\pi}{2n}\sum_{k=1}^{n}(SP_n u)(x_{kn}^\sigma)\widetilde{u}_{n-1}(x_{kn}^\sigma) = \\ &= \frac{1}{2}\langle u, P_n \varphi S \varphi^{-1} L_n^\sigma \widetilde{u}_{n-1}\rangle_\sigma .\end{aligned}$$

Hence,

$$(M_n^\varphi SP_n)^* = P_n \varphi S M_n^\varphi \varphi^{-1} P_n = M_n^\varphi \varphi S M_n^\varphi \varphi^{-1} P_n \tag{3.14}$$

and

$$(M_n^\sigma SP_n)^* = \frac{1}{2} P_n \varphi S \varphi^{-1} L_n^\sigma (P_{n-1} + P_n) = \frac{1}{2}\varphi S \varphi^{-1} L_n^\sigma (P_{n-1} + P_n), \tag{3.15}$$

where in (3.14) we took into account (3.6) and (3.9) and in (3.15) the relations

$$S\varphi^{-1} T_n = -iU_{n-1}, \quad n = 0, 1, 2, \ldots, \quad U_{-1} \equiv 0 . \tag{3.16}$$

In combination with Lemma 3.2 and Corollary 3.3 it is clear now that the sequence $\{(M_n^\omega(aI + bS)P_n)^* P_n\}$ converges strongly on $\mathbf{L}^2_\sigma$ to $\overline{a}I + \varphi S \varphi^{-1}\overline{b}I$ in both cases $\omega = \sigma$ and $\omega = \varphi$.

Step 4: Convergence of $\{W_n M_n^\omega (aI + bS) W_n\}$. We are going to verify the convergence of $W_n M_n^\omega a W_n \widetilde{u}_m$ and $W_n M_n^\omega S W_n \widetilde{u}_m$ for each fixed $m \ge 0$. Let $n > m$. With the help of (3.7), the identity

$$\widetilde{u}_{n-1-m}(x_{kn}^\varphi) = \sqrt{2/\pi}\sin\frac{(n-m)k\pi}{n+1} = (-1)^{k+1}\widetilde{u}_m(x_{kn}^\varphi),$$

and Corollary 3.3 we get

$$\begin{aligned}W_n M_n^\varphi a W_n \widetilde{u}_m &= \sum_{j=0}^{n-1}\alpha_{n-1-j,n}^\varphi(a\widetilde{u}_{n-1-m})\widetilde{u}_j = \\ &= \sum_{j=0}^{n-1}\frac{\pi}{n+1}\sum_{k=1}^{n} a(x_{kn}^\varphi)\widetilde{u}_{n-1-m}(x_{kn}^\varphi)\widetilde{u}_{n-1-j}(x_{kn}^\varphi)\widetilde{u}_j = M_n^\varphi a \widetilde{u}_m .\end{aligned}$$

Consequently,

$$W_nM_n^\varphi aW_n = M_n^\varphi aP_n \longrightarrow aI \quad \text{in} \quad \mathbf{L}^2_\sigma\,. \tag{3.17}$$

To describe the strong limit of $W_nM_n^\sigma aW_n$, we take into account that

$$\widetilde{u}_{n-1-m}(x^\sigma_{kn}) = \sqrt{2/\pi}\sin\frac{(n-m)(2k-1)\pi}{2n} = \sqrt{2/\pi}\sin\frac{2k-1}{2}\pi\cos\frac{m(2k-1)\pi}{2n}$$

i. e.

$$\widetilde{u}_{n-1-m}(x^\sigma_{kn}) = (-1)^{k+1}\gamma_m T_m(x^\sigma_{kn}) \tag{3.18}$$

and

$$L_n^\sigma f = \sum_{j=0}^{n-1}\widetilde{\alpha}^\sigma_{jn}(f)T_j \quad \text{with} \quad \widetilde{\alpha}^\sigma_{jn} = \frac{\pi}{n}\sum_{k=1}^{n} f(x^\sigma_{kn})T_j(x^\sigma_{kn})\,.$$

Then, from (3.11) and Lemma 3.2, we conclude

$$\begin{aligned}
W_nM_n^\sigma aW_n\widetilde{u}_m = &= \sum_{j=0}^{n-1}\alpha^\sigma_{n-1-j,n}(a\widetilde{u}_{n-1-m})\widetilde{u}_j = \\
&= \sum_{j=0}^{n-1}\varepsilon_{n-1-j,n}\frac{\pi}{n}\sum_{k=1}^{n}a(x^\sigma_{kn})\widetilde{u}_{n-1-m}(x^\sigma_{kn})\widetilde{u}_{n-1-j}(x^\sigma_{kn})\widetilde{u}_j = \\
&= \sum_{j=0}^{n-1}\varepsilon_{n-1-j,n}\frac{\pi}{n}\sum_{k=1}^{n}a(x^\sigma_{kn})\gamma_m T_m(x^\sigma_{kn})\gamma_j T_j(x^\sigma_{kn})\widetilde{u}_j = \\
&= \sum_{j=0}^{n-1}\frac{\pi}{n}\sum_{k=1}^{n}a(x^\sigma_{kn})(J_\sigma\widetilde{u}_m)(x^\sigma_{kn})T_j(x^\sigma_{kn})J_\sigma^{-1}T_j = \\
&= J_\sigma^{-1}L_n^\sigma aJ_\sigma\widetilde{u}_m \longrightarrow J_\sigma^{-1}aJ_\sigma\widetilde{u}_m \quad \text{in} \quad \mathbf{L}^2_\sigma\,,
\end{aligned}$$

where J_σ is the isometry introduced in (3.2). Thus,

$$W_nM_n^\sigma aW_n = J_\sigma^{-1}L_n^\sigma aJ_\sigma P_n \longrightarrow J_\sigma^{-1}aJ_\sigma \quad \text{in} \quad \mathbf{L}^2_\sigma\,. \tag{3.19}$$

For the strong convergence of the sequences related with the singular integral observe that, due to (3.6), for all $n > \max\{m,k\}$,

$$\begin{aligned}
\langle W_nM_n^\varphi SW_n\widetilde{u}_m, \widetilde{u}_k\rangle_\sigma &= \langle M_n^\varphi S\widetilde{u}_{n-1-m}, \widetilde{u}_{n-1-k}\rangle_\sigma = i\langle L_n^\varphi\varphi^{-1}T_{n-m}, U_{n-1-k}\rangle_\varphi = \\
&= \frac{2i}{n+1}\sum_{j=1}^{n}\cos\frac{(n-m)j\pi}{n+1}\,\sin\frac{(n-k)j\pi}{n+1} = \\
&= -i\langle L_n^\varphi\varphi^{-1}T_{m+1}, U_k\rangle_\varphi = -\langle M_n^\varphi SP_n\widetilde{u}_m, \widetilde{u}_k\rangle_\sigma\,.
\end{aligned}$$

Hence,

$$W_nM_n^\varphi SW_n = -M_n^\varphi SP_n \longrightarrow -S \quad \text{in} \quad \mathbf{L}^2_\sigma\,. \tag{3.20}$$

Further, the identities

$$J_\sigma(\widetilde{u}_{n+2} - \widetilde{u}_n) = \gamma_{n+2}T_{n+2} - \gamma_n T_n = -2\varphi\widetilde{u}_n\,, \quad n \geq 0\,, \tag{3.21}$$

(3.6), (3.9), and (3.19) imply for $n > m \geq 1$, (in case $n = m-1$, note that $L_n^\sigma T_n = 0$)

$$\begin{aligned} W_n M_n^\sigma S W_n \widetilde{u}_m &= \frac{i}{2} W_n M_n^\sigma \varphi^{-1} W_n(\widetilde{u}_{m-1} - \widetilde{u}_{m+1}) = -\frac{i}{2} J_\sigma^{-1} L_n^\sigma \varphi^{-1} J_\sigma(\widetilde{u}_{m+1} - \widetilde{u}_{m-1}) = \\ &= i J_\sigma^{-1} L_n^\sigma \widetilde{u}_{m-1} \longrightarrow i J_\sigma^{-1} \widetilde{u}_{m-1} \quad \text{in} \quad \mathbf{L}_\sigma^2 . \end{aligned}$$

Obviously, $W_n M_n^\sigma S W_n \widetilde{u}_0 = i W_n M_n^\sigma T_n = 0$. Hence, by means of the shift operator V introduced in (3.3) we can formulate the derived convergence result as follows:

$$W_n M_n^\sigma S W_n = i J_\sigma^{-1} L_n^\sigma V^* P_n \longrightarrow i J_\sigma^{-1} V^* \quad \text{in} \quad \mathbf{L}_\sigma^2 . \tag{3.22}$$

Step 5: Convergence of $\{(W_n M_n^\omega (aI + bS) W_n)^*\}$. In case $\omega = \varphi$, the strong convergence of this sequence follows from (3.17), (3.20), (3.12) and (3.14), together with the outcome of step 3. In case $\omega = \sigma$ we have, in view of (3.19),

$$\begin{aligned} \langle W_n M_n^\sigma a W_n u, v\rangle_\sigma &= \langle L_n^\sigma a J_\sigma P_n u, J_\sigma^{-*} P_n v\rangle_\sigma = \frac{\pi}{n} \sum_{j=1}^n a(x_{jn}^\sigma)(J_\sigma P_n u)(x_{jn}^\sigma)\overline{(J_\sigma^{-*} P_n v)(x_{jn}^\sigma)} = \\ &= \langle u, J_\sigma^* L_n^\sigma \overline{a} J_\sigma^{-*} P_n v\rangle_\sigma \end{aligned}$$

i. e. $(W_n M_n^\sigma a W_n)^* = J_\sigma^* L_n^\sigma \overline{a} J_\sigma^{-*} P_n \longrightarrow J_\sigma^* \overline{a} J_\sigma^{-*}$ in $\mathbf{L}_\sigma^2$. Using (3.22), we get in the same manner

$$\begin{aligned} \langle W_n M_n^\sigma S W_n u, v\rangle_\sigma &= i\langle L_n^\sigma V^* P_n u, J_\sigma P_n v\rangle_\sigma = \frac{\pi i}{n} \sum_{j=1}^n (V^* P_n u)(x_{jn}^\sigma)\overline{(J_\sigma^{-*} P_n v)(x_{jn}^\sigma)} = \\ &= i\langle \varphi V^* P_n u, L_n^\sigma \varphi^{-1} J_\sigma^{-*} P_n\rangle_\sigma = i\langle u, V M_n^\sigma J_\sigma^{-*} P_n\rangle_\sigma , \end{aligned}$$

whence the strong convergence of $(W_n M_n^\sigma S W_n)^*$.

For further considerations we need Fredholm and invertibility conditions for the operator $aI + bS : \mathbf{L}_\sigma^2 \longrightarrow \mathbf{L}_\sigma^2$, if $a, b \in \mathbf{PC}$. For this goal, we define $c := (a+b)/(a-b)$ on $[-1, 1]$ and $f(\mu) := \exp(i\pi(\mu-1)/2)\sin(\pi\mu/2)$ on $[0, 1]$, and we associate to this operator the function

$$\mathbf{c}(x, \mu) := \begin{cases} c(x)(1-\mu) + c(x+0)\mu , & \mu \in [0,1] ,\ x \in (-1,1) , \\ c(1) + [1 - c(1)] f(\mu) , & \mu \in [0,1] ,\ x = 1 , \\ 1 + [c(-1) - 1] f(\mu) , & \mu \in [0,1] ,\ x = -1 . \end{cases}$$

Note that, for $z_1, z_2 \in \mathbb{C}$, $z_1 + (z_2 - z_1) f(\mu)$, $\mu \in [0,1]$, describes the half circle from z_1 to z_2 that lies to the right of the straight line from z_1 to z_2. Thus, if $c(x \pm 0)$ is finite for all $x \in [-1,1]$, the image of $\mathbf{c}(x,\mu)$ is a closed curve in the complex plane which possesses a natural orientation, and by $\operatorname{wind} \mathbf{c}(x,\mu)$ we denote the winding number of this curve w.r.t. the origin 0.

Lemma 3.5 ([6], Theorem IX.4.1).. *Let $a, b \in \mathbf{PC}$. The operator $A = aI + bS : \mathbf{L}_\sigma^2 \longrightarrow \mathbf{L}_\sigma^2$ is Fredholm if and only if $a(x \pm 0) - b(x \pm 0) \neq 0$ for all $x \in [-1,1]$ and $\mathbf{c}(x,\mu) \neq 0$ for all $(x,\mu) \in [-1,1] \times [0,1]$. In this case, A is one-sided invertible and* $\operatorname{ind} A = -\operatorname{wind} \mathbf{c}(x,\mu)$.

Lemma 3.6. *Let $a, b \in \mathbf{PC}$. Then the operator $\widetilde{W}\{A_n^\sigma\} = J_\sigma^{-1}(aJ_\sigma + ibV^*)$ is invertible in $\mathbf{L}_\sigma^2$ if and only if the operator $W\{A_n^\sigma\} = aI + bS$ is invertible in $\mathbf{L}_\sigma^2$.*

Proof. The invertibility of $\widetilde{W}\{A_n^\sigma\}$ is equivalent to the invertibility of $B = aJ_\sigma + ibV^*$. Since $J_\sigma = \varphi I - i\psi S$ and $V^* = \psi I + i\varphi S$ with $\psi(x) = x$ (this follows from (3.6), (3.9), and (3.8)), the operator B is again a singular integral operator. Thus, the invertibility of B is equivalent to the Fredholmness of B with index 0, or to the Fredholmness of $C = BV$ with index -1. With the help of $V = \psi I - i\varphi S$ and $S\varphi S = \varphi I + K_0$, where $K_0 u = -1/\sqrt{2\pi}\langle u, \widetilde{u}_0\rangle_\sigma$ (see (4.3) below), we get

$$C = a(\varphi I - i\psi S)(\psi I - i\varphi S) + ibI = -ia\varphi^2 S - i\psi^2 S + ibI + K = i(bI - aS) + K\,,$$

with a compact operator $K : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$. Now the assertion follows from $b - a/b + a = -\left(a + b/a - b\right)^{-1}$ and Lemma 3.5.

4. Local theory of stability

Let $\mathcal{A}^\omega$ denote the smallest C^*-subalgebra of $\mathcal{F}^W$ which contains all sequences of the form $\{M_n^\omega(aI + bS)P_n\}$ with $a, b \in \mathbf{PC}$ and the ideal $\mathcal{J}$. The aim of this section is to derive necessary and sufficient conditions for the stability of the sequences $\{M_n^\omega(aI + bS)P_n\}$ and, more general, for arbitrary sequences $\{A_n\} \in \mathcal{A}^\omega$. Our approach to these results is essentially based on the application of the local principles by Allan/Douglas and Gohberg/Krupnik to study the invertibility of a coset $\{A_n\}^o := \{A_n\} + \mathcal{J}$ in the quotient algebra $\mathcal{A}^\omega/\mathcal{J}$. In what follows we agree upon omitting the superscript ω in all notations (such as in M_n^ω, which will be abbreviated to M_n) whenever the validity of the assertion where this notation is used does not depend on $\omega = \varphi$ or $\omega = \sigma$.

The applicability of a local principle in a Banach algebra depends on the existence of sufficiently many elements which commute with every other element of the algebra, i. e. which belong to the center of the algebra. The following lemma establishes the existence of such elements for the algebra $\mathcal{A}^\omega/\mathcal{J}$.

Lemma 4.1. *If $f \in \mathbf{C}[-1, 1]$, then the coset $\{M_n f P_n\}^o$ commutes with every coset $\{A_n\}^o \in \mathcal{A}^\omega/\mathcal{J}$.*

Proof. It is enough to verify that $\{M_n f P_n\}^o$ commutes with all cosets $\{M_n a P_n\}^o$ where $a \in \mathbf{PC}$ and with the coset $\{M_n S P_n\}^o$. The first assertion is obvious; one even has $\{M_n a P_n\}\{M_n f P_n\} = \{M_n f P_n\}\{M_n a P_n\}$ for arbitrary a, $f \in \mathbf{PC}$. For the second assertion, note that the equalities $M_n f P_n M_n S P_n = M_n f S P_n$ and $M_n S P_n M_n f P_n = M_n S M_n f P_n$ hold. So, what remains to prove is

$$\{M_n f S P_n - M_n S M_n f P_n\} \in \mathcal{J} \quad \text{for all} \quad f \in \mathbf{C}[-1, 1]\,. \tag{4.1}$$

We show that (4.1) holds true for all algebraic polynomials p in place of f. Then the assertion follows from the closedness of $\mathcal{J}$ and from (3.4) and (3.5). So let p be a polynomial of degree not greater than m. Then $M_n p P_{n-m} = p P_{n-m}$ for $n > m$. Consequently,

$$M_n p S P_n - M_n S M_n p P_n = M_n(pS - Sp)P_n + M_n S(I - M_n)p(P_n - P_{n-m})\,.$$

Obviously, the sequence $\{M_n(pS - Sp)P_n\}$ belongs to $\mathcal{J}$. Moreover, $P_n - P_{n-m} = W_n P_m W_n$, which implies that

$$\begin{aligned} M_n S(I - M_n)p(P_n - P_{n-m}) &= M_n S(I - M_n)p P_n W_n P_m W_n = \\ &= \Big(M_n(Sp - pS)P_n + M_n p S P_n - M_n S P_n M_n p P_n\Big) W_n P_m W_n \end{aligned}$$

and, hence, $\{M_n S(I - M_n)p(P_n - P_{n-m})\} \in \mathcal{J}$.

Together with the identities $M_n f_1 M_n f_2 P_n = M_n f_1 f_2 P_n$ and $(M_n f P_n)^* = M_n \overline{f} P_n$, the preceding lemma shows that the set $\mathcal{C}^\omega := \{\{M_n f P_n\}^o : f \in \mathbf{C}[-1,1]\}$ forms a C^*-subalgebra of the center of $\mathcal{A}^\omega/\mathcal{J}$. This offers the applicability of the local principle by Allan/Douglas which we recall here for the reader's convenience.

Local principle of Allan and Douglas (comp. [1, 4]). Let $\mathcal{B}$ be a unital Banach algebra and let $\mathcal{B}_c$ be a closed subalgebra of the center of $\mathcal{B}$ containing the identity. For every maximal ideal $x \in M(\mathcal{B}_c)$, let $\mathcal{J}_x$ denote the smallest closed ideal of $\mathcal{B}$ which contains x, i. e.

$$\mathcal{J}_x = \mathrm{clos}_{\mathcal{B}} \left\{ \sum_{j=1}^m a_j c_j : a_j \in \mathcal{B},\, c_j \in x,\, m = 1, 2, \ldots \right\}.$$

Then an element $a \in \mathcal{B}$ is invertible in $\mathcal{B}$ if and only if $a + \mathcal{J}_x$ is invertible in $\mathcal{B}/\mathcal{J}_x$ for all $x \in M(\mathcal{B}_c)$. (In case $\mathcal{J}_x = \mathcal{B}$ we define that $a + \mathcal{J}_x$ is invertible.) Moreover, the mapping

$$M(\mathcal{B}_c) \longrightarrow [0, \infty), \quad x \mapsto \|b + \mathcal{J}_x\|$$

is upper semi-continuous for each $b \in \mathcal{B}$. In case $\mathcal{B}$ is a C^*-algebra and $\mathcal{B}_c$ a is central C^*-subalgebra of $\mathcal{B}$, then all ideals $\mathcal{J}_x$ are proper ideals of $\mathcal{B}$, and $\|b\| = \max\{\|b + \mathcal{J}_x\| : x \in M(\mathcal{B}_c)\}$ for all $b \in \mathcal{B}$.

We will apply this local principle with $\mathcal{A}^\omega/\mathcal{J}$ and $\mathcal{C}^\omega$ in place of $\mathcal{B}$ and $\mathcal{B}_c$, respectively. The algebra $\mathcal{C}^\omega$ is *-isomorphic to $\mathbf{C}[-1,1]$ via the isomorphism $\{M_n f P_n\}^o \mapsto f$. This can be seen as follows: If $f \in \mathbf{C}[-1,1]$ is invertible, then the coset $\{M_n f P_n\}^o$ is invertible. Conversely, assume this coset is invertible and choose one of its representatives $\{M_n f P_n + P_n K P_n + W_n L W_n + G_n\}$ which then is invertible modulo $\mathcal{J}$. An application of the homomorphism W yields the invertibility of $fI + K$ modulo compact operators, i. e. the Fredholmness of the multiplication operator fI. But Fredholm multiplication operators are invertible.

Consequently, the maximal ideal space of $\mathcal{C}$ is equal to $\{\mathcal{I}_\tau^\omega : \tau \in [-1,1]\}$ with

$$\mathcal{I}_\tau^\omega := \left\{ \{M_n^\omega f P_n\}^o : f \in \mathbf{C}[-1,1],\ f(\tau) = 0 \right\}.$$

Let $\mathcal{J}_\tau^\omega$ denote the smallest closed ideal of $\mathcal{A}^\omega/\mathcal{J}$ which contains the maximal ideal $\mathcal{I}_\tau^\omega$ of $\mathcal{C}^\omega$, i. e.

$$\mathcal{J}_\tau^\omega = \mathrm{clos}_{\mathcal{A}^\omega/\mathcal{J}} \left\{ \sum_{j=1}^m \{A_n^j M_n^\omega f_j P_n\}^o : \{A_n^j\} \in \mathcal{A}^\omega, f_j \in \mathbf{C}[-1,1], f_j(\tau) = 0,\, m = 1, 2, \ldots \right\}.$$

Then the local principle of Allan/Douglas says that all ideals $\mathcal{J}_\tau^\omega$ are proper in $\mathcal{A}^\omega/\mathcal{J}$, and that a coset $\{A_n\}^o$ is invertible in $\mathcal{A}^\omega/\mathcal{J}$ if and only if $\{A_n\}^o + \mathcal{J}_\tau$ is invertible in $(\mathcal{A}^\omega/\mathcal{J})/\mathcal{J}_\tau^\omega$ for every $\tau \in [-1,1]$.

Our next goal is the description of the local algebras $(\mathcal{A}^\omega/\mathcal{J})/\mathcal{J}_\tau^\omega$. First let $-1 < \tau < 1$. Let h_τ be the function which is 0 on $[-1, \tau]$ and 1 on $(\tau, 1]$. Then, for every $a \in \mathbf{PC}$,

$$\{M_n a P_n\}^o + \mathcal{J}_\tau = a(\tau + 0)\{M_n h_\tau P_n\}^o + a(\tau)\{M_n(1 - h_\tau)P_n\}^o + \mathcal{J}_\tau.$$

Consequently, the algebra $(\mathcal{A}^\omega/\mathcal{J})/\mathcal{J}_\tau^\omega$ is generated by its cosets $e := \{P_n\}^o + \mathcal{J}_\tau$,

$$p := \frac{1}{2}\left(\{P_n\}^o + \{M_n S P_n\}^o\right) + \mathcal{J}_\tau, \quad \text{and} \quad q := \{M_n h_\tau P_n\}^o + \mathcal{J}_\tau. \tag{4.2}$$

Obviously, q is a selfadjoint projection. In order to see that the same is true for p, we make use of the relation

$$S\varphi S = \varphi I + K_0\,, \quad \text{where} \quad K_0 u = -\frac{1}{\sqrt{2}}\,\langle u, \widetilde{u}_0\rangle_\sigma T_0\,, \tag{4.3}$$

which is a consequence of (3.6), (3.9), (3.21) and of the continuity of the operator $S\varphi S : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$. Indeed,

$$\begin{aligned} S\varphi S\widetilde{u}_n &= iS\varphi T_{n+1} = \frac{i}{2}\,S\varphi(U_{n+1} - U_{n-1}) = \\ &= \begin{cases} \frac{1}{2}(T_n - T_{n+2}), & n \ge 1, \\ -\frac{1}{2}\,T_2, & n = 0\,, \end{cases} = \begin{cases} \varphi\widetilde{u}_n, & n \ge 1, \\ \varphi\widetilde{u}_0 - \frac{1}{\sqrt{2}}\,T_0, & n = 0\,. \end{cases} \end{aligned}$$

Further we recall that $SP_n u = i\sum_{k=0}^{n-1}\langle u, \widetilde{u}_k\rangle_\sigma T_{k+1} = \frac{i}{2}\sum_{k=0}^{n-1}\langle u, \widetilde{u}_k\rangle_\sigma (U_{k+1} - U_{k-1})$, which implies

$$M_n^\varphi \varphi SP_n = \varphi SP_n - \frac{i}{2} VW_nP_1W_n\,. \tag{4.4}$$

Consequently, we have the identities $M_n^\varphi SP_nM_n^\varphi \varphi SP_n = M_n^\varphi S\varphi SP_n - \frac{i}{2}M_n^\varphi SVP_nW_nP_1W_n$ and $M_n^\sigma \varphi SP_n = \varphi SP_n - i\varphi J_\sigma VW_nP_1W_n$. Thus, in both cases,

$$\begin{aligned} \{M_nSP_n\}^o\{M_nSP_n\}^o + \mathcal{J}_\tau &= \frac{1}{\varphi(\tau)}\,\{M_nSP_n\}^o\{M_n\varphi SP_n\}^o + \mathcal{J}_\tau = \\ &= \frac{1}{\varphi(\tau)}\,\{M_n\varphi P_n\}^o + \mathcal{J}_\tau = \{P_n\}^o + \mathcal{J}_\tau\,,\ -1 < \tau < 1\,. \end{aligned} \tag{4.5}$$

The identities (3.6), (3.8), and (3.9) imply that, with $\psi(x) = x$,

$$V = \psi I - i\varphi S\,, \quad V^* = \psi I + i\varphi S\,. \tag{4.6}$$

From this we can conclude that $\{M_nSP_n\}^o + \mathcal{J}_\tau$ is selfadjoint. Indeed,

$$\{M_nSP_n\}^o + \mathcal{J}_\tau = -\frac{i}{\varphi(\tau)}\{M_n i\varphi SP_n\}^o + \mathcal{J}_\tau = -\frac{i}{\varphi(\tau)}\,(\{V^*P_n\}^o - \{M_n\psi P_n\}^o) + \mathcal{J}_\tau$$

and, consequently,

$$\begin{aligned} (\{M_nSP_n\}^o + \mathcal{J}_\tau)^* &= \frac{i}{\varphi(\tau)}\,(\{P_nVP_n\}^o - \{M_n\psi P_n\}^o) + \mathcal{J}_\tau = \\ &= \frac{i}{\varphi(\tau)}\{M_n(-i\varphi)SP_n\}^o + \mathcal{J}_\tau = \{M_nSP_n\}^o + \mathcal{J}_\tau\,. \end{aligned}$$

So we have seen that the local algebra $(\mathcal{A}^\omega/\mathcal{J})/\mathcal{J}^\omega_\tau$ is generated by its identity element and by two projections in case $-1 < \tau < 1$. Algebras of this kind are described by the following result.

Theorem 4.2 (Halmos' two-projections theorem, [8]).. *Let $\mathcal{B}$ be a unital C^*-algebra, and let $p, q \in \mathcal{B}$ be projections (i. e. self-adjoint idempotent elements) such that $\sigma_{\mathcal{B}}(pqp) = [0,1]$. Then the smallest closed subalgebra of $\mathcal{B}$, which contains p, q, and the identity element e, is *-isomorphic to the C^*-algebra of all continuous 2×2 matrix functions on $[0,1]$, which are diagonal at 0 and 1. The isomorphism can be chosen in such a way that it sends e, p, and q into the functions*

$$\mu \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mu \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \mu \mapsto \begin{bmatrix} \mu & \sqrt{\mu(1-\mu)} \\ \sqrt{\mu(1-\mu)} & 1-\mu \end{bmatrix}, \tag{4.7}$$

respectively.

To apply this theorem, we have to check whether $\sigma_{(\mathcal{A}/\mathcal{J})/\mathcal{J}_\tau}(pqp) = [0,1]$ for p and q defined by (4.2). For this, let $\mathcal{G}$ be the smallest C^*-subalgebra of $\mathcal{L}(\mathbf{L}^2_\sigma)$ which contains all operators $aI + bS$ with $a, b \in \mathbf{PC}[-1,1]$ and the ideal $\mathcal{K} = \mathcal{K}(\mathbf{L}^2_\sigma)$ of all compact operators on $\mathbf{L}^2_\sigma$. By $\mathcal{J}^{\mathcal{G}}_\tau$, $\tau \in [-1,1]$, we denote the smallest closed ideal of $\mathcal{G}/\mathcal{K}$, which contains all cosets $fI + \mathcal{K}$ with $f \in \mathbf{C}[-1,1]$ and $f(\tau) = 0$.

Lemma 4.3. *If $\{A_n\}^o + \mathcal{J}_\tau$ is invertible in $(\mathcal{A}/\mathcal{J})/\mathcal{J}_\tau$, then $(W\{A_n\} + \mathcal{K}) + \mathcal{J}^{\mathcal{G}}_\tau$ is invertible in $(\mathcal{G}/\mathcal{K})/\mathcal{J}^{\mathcal{G}}_\tau$.*

Proof. Let $\{A_n\} \in \mathcal{A}$, and assume that there is a sequence $\{B_n\} \in \mathcal{A}$ such that $\{B_n\}^o\{A_n\}^o + \mathcal{J}_\tau = \{P_n\}^o + \mathcal{J}_\tau$. Then $B_nA_n = P_n + J_n + P_nKP_n + W_nTW_n + C_n$ with some operators $K, T \in \mathcal{K}$, some coset $\{J_n\}^o \in \mathcal{J}_\tau$ and some sequence $\{C_n\} \in \mathcal{N}$. Further, given $\varepsilon > 0$, there exist sequences $\{A^{(j)}_n\} \in \mathcal{A}$ and functions $f_j \in \mathbf{C}[-1,1]$ with $f_j(\tau) = 0$ such that $\|\{J_n\}^o - \{B'_n\}^o\|_{\mathcal{A}/\mathcal{J}} < \varepsilon$ for $B'_n = \displaystyle\sum_{j=1}^{m_\varepsilon} A^{(j)}_n M_n f_j P_n$. Hence, there are operators $K_\varepsilon, T_\varepsilon \in \mathcal{K}(\mathbf{L}^2_\sigma)$ and a sequence $\{C^\varepsilon_n\} \in \mathcal{N}$ such that

$$\left\| J_nP_n - \sum_{j=1}^{m_\varepsilon} A^{(j)}_n M_n f_j P_n - P_nK_\varepsilon P_n - W_nT_\varepsilon W_n - C^\varepsilon_n P_n \right\|_{\mathcal{L}(\mathbf{L}^2_\sigma)} < \varepsilon \quad \text{for all} \quad n \geq 1.$$

Hence, $\left\| W\{J_n\} - \sum_{j=1}^{m_\varepsilon} W\{A^{(j)}_n\} f_j I - K_\varepsilon \right\|_{\mathcal{L}(\mathbf{L}^2_\sigma)} \leq \varepsilon$, which implies $W\{J_n\} + \mathcal{K} \in \mathcal{J}^{\mathcal{G}}_\tau$. Thus, because of $W\{B_n\}W\{A_n\} = I + W\{J_n\} + K$, the coset $(W\{A_n\} + \mathcal{K}) + \mathcal{J}^{\mathcal{G}}_\tau$ is invertible from the left in $(\mathcal{G}/\mathcal{K})/\mathcal{J}^{\mathcal{G}}_\tau$. Its invertibility from the right can be shown analogously.

Now we can complete the description of the local algebras in case $-1 < \tau < 1$. The product pqp is a non-negative element of $(\mathcal{A}/\mathcal{J})/\mathcal{J}_\tau$, which implies that its spectrum $\sigma_{(\mathcal{A}/\mathcal{J})/\mathcal{J}_\tau}(pqp)$ is contained in $[0,1]$. We prove that the spectrum of pqp coincides with this interval. Assume there is a $\lambda \in (0,1)$ such that $pqp - \lambda e$ is invertible in $(\mathcal{A}/\mathcal{J})/\mathcal{J}_\tau$. The invertibility of $pqp - \lambda e$ is equivalent to the invertibility of

$$(q - \lambda)p - \lambda(e - p) =$$

$$= \frac{1}{2}\{M_n(h_\tau - \lambda)P_n\}^o\left(\{P_n\}^o + \{M_nSP_n\}^o\right) - \frac{\lambda}{2}\left(\{P_n\}^o - \{M_nSP_n\}^o\right) + \mathcal{J}_\tau .$$

Lemma 4.3 implies that $(A + \mathcal{K}) + \mathcal{J}^{\mathcal{G}}_\tau := \Big((h_\tau - \lambda)(I + S) - \lambda(I - S) + \mathcal{K}\Big) + J^{\mathcal{G}}_\tau$ is invertible in $(\mathcal{G}/\mathcal{K})/\mathcal{J}^{\mathcal{G}}_\tau$. If $-1 \leq x < \tau$, we have $(A + \mathcal{K}) + \mathcal{J}^{\mathcal{G}}_x = (-2\lambda I + \mathcal{K}) + \mathcal{J}^{\mathcal{G}}_x$, and $-2\lambda I + \mathcal{K}$

is invertible in $\mathcal{G}/\mathcal{K}$. If $\tau < x \le 1$ then $(A+\mathcal{K}) + \mathcal{J}_x^{\mathcal{G}} = \Big((1-2\lambda)I + S + \mathcal{K}\Big) + \mathcal{J}_x^{\mathcal{G}}$, which is also invertible in $(\mathcal{G}/\mathcal{K})/\mathcal{J}_x^{\mathcal{G}}$. From the local principle of Allan and Douglas we conclude the Fredholmness of $(h_\tau - \lambda)(I+S) - \lambda(I-S)$ in $\mathbf{L}^2_\sigma$. But this is in contradiction to (see Lemma 3.5) $0 \in \left[1 - \frac{1}{\lambda}, 1\right] = \left[\frac{\lambda - h_\tau(\tau+0)}{\lambda}, \frac{\lambda - h_\tau(\tau-0)}{\lambda}\right]$.

Thus we can apply Halmos' two projections theorem to get that the local algebra $(\mathcal{A}/\mathcal{J})/\mathcal{J}_\tau$ is *-isomorphic to the C^*-algebra of the continuous 2×2 matrix functions on $[0,1]$ which are diagonal at 0 and 1. The isomorphism can be chosen in such a way that it sends $\{P_n\}^o + \mathcal{J}_\tau$, $\frac{1}{2}(\{P_n\}^o + \{M_n S P_n\}^o) + \mathcal{J}_\tau$, and $\{M_n h_\tau P_n\}^o + \mathcal{J}_\tau$ into the functions given in (4.7), respectively.

Now we turn our attention to the local algebras at $\tau = \pm 1$. Since $\{M_n(aI+bS)P_n\}^o + \mathcal{J}_\tau = \{M_n[a(\tau)I + b(\tau)S]P_n\}^o + \mathcal{J}_\tau$, these local algebras are generated (as C^*-algebras) by their cosets $\{P_n\}^o + \mathcal{J}_\tau$ (the identity element) and $\{M_n S P_n\}^o + \mathcal{J}_\tau$. It turns out that the properties of the latter coset (and, thus, the behaviour of the algebras generated by it) depends heavily on the weight function ω. For $\omega = \sigma$ we have the following result, the proof of which will be given in the Appendix.

Lemma 4.4. *Let $\tau = \pm 1$. The coset $\{M_n^\sigma S P_n\}^o + \mathcal{J}_\tau$ is a unitary element of the algebra $(\mathcal{A}^\sigma/\mathcal{J})/\mathcal{J}_\tau^\sigma$, and its spectrum is equal to $\mathbb{T}_\tau$, where $\mathbb{T}_{\pm 1} = \mathbb{T} \cap \{t \in \mathbb{C} : \pm \Im t \ge 0\}$.*

Consequently, the algebra $(\mathcal{A}^\sigma/\mathcal{J})/\mathcal{J}_\tau^\sigma$ is *-isomorphic to the algebra $\mathbf{C}(\mathbb{T}_\tau)$ of all complex valued continuous functions on $\mathbb{T}_\tau$, and the isomorphism can be chosen such that it sends $\{M_n^\sigma S P_n\}^o + \mathcal{J}_\tau^\sigma$ into the function $t \mapsto t$.

The treatment of the case $\omega = \varphi$ starts with the following lemma.

Lemma 4.5. *The sequences $\{M_n^\varphi \varphi^{-1} P_n\}$ and $\{(M_n^\varphi \varphi^{-1} P_n)^*\}$ converge strongly to the multiplication operators $\varphi^{-1} I : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\varphi$ and $\varphi I : \mathbf{L}^2_\varphi \longrightarrow \mathbf{L}^2_\sigma$, respectively.*

Proof. *Convergence of $M_n^\varphi \varphi^{-1} P_n$:* Since $\mathbf{L}^2_\sigma$ is continuously embedded into $\mathbf{L}^2_\varphi$ we have, due to Corollary 3.3,

$$\lim_{n\to\infty} M_n^\varphi \varphi^{-1} P_n \widetilde{u}_m = \lim_{n\to\infty} M_n^\varphi U_m = U_m = \varphi^{-1}\widetilde{u}_m \quad \text{for all} \quad m \ge 0 \quad \text{in} \quad \mathbf{L}^2_\varphi .$$

Thus, it remains to show that the operators $M_n^\varphi \varphi^{-1} P_n : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\varphi$ are uniformly bounded. Consider the quadrature rule

$$\widetilde{Q}_n f = \int_{-1}^1 (L_n^\varphi f)(x)\rho(x)\,dx = \sum_{k=1}^n \rho_{kn} f(x_{kn}^\varphi),$$

where $\rho(x) = (1-x^2)^{3/2}$, and abbreviate x_{kn}^φ to x_k. The quadrature weights ρ_{kn} are equal to

$$\begin{aligned}
\rho_{kn} &= \int_{-1}^1 \frac{U_n(x)}{x-x_k}\,\frac{(1-x^2)\varphi(x)}{U_n'(x_k)}\,dx = \\
&= (1-x_k^2)\int_{-1}^1 \frac{U_n(x)\varphi(x)\,dx}{(x-x_k)U_n'(x_k)} - \frac{1}{U_n'(x_k)}\int_{-1}^1 U_n(x)(x+x_k)\varphi(x)\,dx = \\
&= (1-x_k^2)\lambda_{kn}^\varphi, \quad n > 1.
\end{aligned}$$

Define $\varepsilon_{ij} := \langle \ell_{in}^\varphi, \ell_{jn}^\varphi \rangle_\rho$ for $i, j = 1, \ldots, n$. We remark that $U_n'(x_k) = \sqrt{2/\pi}\dfrac{(-1)^{k+1}(n+1)}{1-x_k^2}$ and compute

$$\begin{aligned}\eta_k &:= \int_{-1}^1 \frac{[U_n(x)]^2}{x-x_k}\,\rho(x)\,dx = \\ &= (1-x_k^2)\int_{-1}^1 \frac{[U_n(x)]^2}{x-x_k}\,\varphi(x)\,dx - \int_{-1}^1 (x+x_k)\,[U_n(x)]^2\,\varphi(x)\,dx = -x_k\,,\end{aligned}$$

where we take into account the orthogonality and symmetry properties of U_n. For $i \neq j$, it follows

$$\begin{aligned}\varepsilon_{ij} &= \frac{\pi(-1)^{i+j}(1-x_i^2)(1-x_j^2)}{2(n+1)^2}\int_{-1}^1 \frac{[U_n(x)]^2\,\rho(x)\,dx}{(x-x_i)(x-x_j)} = \\ &= \frac{\pi(-1)^{i+j}(1-x_i^2)(1-x_j^2)}{2(n+1)^2}\,\frac{\eta_i-\eta_j}{x_i-x_j} = \frac{\pi(-1)^{i+j+1}(1-x_i^2)(1-x_j^2)}{2(n+1)^2}\,,\end{aligned}$$

i. e.

$$\operatorname{sgn}\varepsilon_{ij} = (-1)^{i+j+1} \quad \text{for} \quad i \neq j\,. \tag{4.8}$$

For $i = j$, we get

$$\begin{aligned}[U_n'(x_j)]^2\,\varepsilon_{jj} &= \int_{-1}^1 \frac{[U_n(x)]^2\,\rho(x)\,dx}{(x-x_j)^2} = \\ &= (1-x_j^2)\int_{-1}^1 \frac{[U_n(x)]^2}{(x-x_j)^2}\,\varphi(x)\,dx - \int_{-1}^1 \frac{[U_n(x)]^2}{x-x_j}(x+x_j)\varphi(x)\,dx = \\ &= (1-x_j^2)\lambda_{jn}^\varphi\,[U_n'(x_j)]^2 + 1 - 2\int_{-1}^1 \frac{[U_n(x)]^2}{x-x_j}\,x\varphi(x)\,dx = \\ &= (1-x_j^2)\lambda_{jn}^\varphi\,[U_n'(x_j)]^2 + 1 - \\ &\qquad - \int_{-1}^1 [U_{n+1}(x)+U_{n-1}(x)]\,\frac{U_n(x)}{x-x_j}\,\varphi(x)\,dx = \\ &= (1-x_j^2)\lambda_{jn}^\varphi\,[U_n'(x_j)]^2 + 1 - \lambda_{jn}^\varphi U_{n-1}(x_j)U_n'(x_j)\,.\end{aligned}$$

Since $U_{n-1}(x_j) = \sqrt{2/\pi}\dfrac{\sin(nj\pi/n+1)}{\sin(j\pi/n+1)} = \sqrt{2/\pi}(-1)^{j+1}$, we have $\lambda_{jn}^\varphi U_{n-1}(x_j)U_n'(x_j) = 2$, which yields

$$\varepsilon_{jj} < (1-x_j^2)\lambda_{jn}^\varphi = \rho_{jn}\,, \quad j = 1, \ldots, n\,. \tag{4.9}$$

Let now $f:(-1,1)\longrightarrow\mathbb{C}$ be given. Then, due to (4.8) and (4.9),

$$\begin{aligned}\int_{-1}^{1}\left|(L_n^\varphi f)(x)\right|^2\rho(x)\,dx &= \sum_{i=1}^n\sum_{j=1}^n f(x_i)\overline{f(x_j)}\varepsilon_{ij}\le\sum_{i=1}^n\sum_{j=1}^n|f(x_i)|\,|f(x_j)|\,|\varepsilon_{ij}|\le\\ &\le 2\sum_{i=1}^n|f(x_i)|^2\varepsilon_{ii}-\sum_{i=1}^n\sum_{j=1}^n(-1)^{i+j}|f(x_i)|\,|f(x_j)|\varepsilon_{ij}\le\\ &\le 2\sum_{i=1}^n\rho_{in}|f(x_i)|^2-\int_{-1}^1\sum_{i=1}^n\left[(-1)^i|f(x_i)|\ell_{in}^\varphi(x)\right]^2\rho(x)\,dx\le\\ &\le 2\widetilde{Q}_n|f|^2\,.\end{aligned}$$

Using this estimate in combination with the explicit form of the quadrature weights ρ_{kn} derived above we obtain, for $u_n=\varphi v_n\in\operatorname{im}P_n$,

$$\begin{aligned}\left\|M_n^\varphi\varphi^{-1}u_n\right\|_\varphi^2 &= \left\|\varphi L_n^\varphi\varphi^{-1}v_n\right\|_\varphi^2=\left\|L_n^\varphi\varphi^{-1}v_n\right\|_\rho^2\le\\ &\le 2\widetilde{Q}_n\left|\varphi^{-1}v_n\right|^2=2\sum_{k=1}^n\lambda_{kn}^\varphi|v_n(x_{kn}^\varphi)|^2=2\left\|v_n\right\|_\varphi^2=2\left\|u_n\right\|_\sigma^2\,,\end{aligned}$$

which proves the desired uniform boundedness.

Convergence of $(M_n^\varphi\varphi^{-1}P_n)^*$: The strong convergence of $M_n^\varphi\varphi^{-1}P_n$ implies the uniform boundedness of $(M_n^\varphi\varphi^{-1}P_n)^*:\mathbf{L}_\varphi^2\longrightarrow\mathbf{L}_\sigma^2$. Since $\{\varphi^{-1}T_m\}_{m=0}^\infty$ forms an orthonormal basis in $\mathbf{L}_\varphi^2$, it remains to prove that $(M_n^\varphi\varphi^{-1}P_n)^*\varphi^{-1}T_m\longrightarrow T_m$ in $\mathbf{L}_\sigma^2$. In view of

$$\eta_{nmj}:=\langle(M_n^\varphi\varphi^{-1}P_n)^*\varphi^{-1}T_m,\widetilde{u}_j\rangle_\sigma=\langle\varphi^{-1}T_m,M_n^\varphi\varphi^{-1}P_n\widetilde{u}_j\rangle_\varphi,$$

we have $\eta_{nmj}=0$ for $j\ge n$ and, for $n>m$ and $j<n$,

$$\eta_{nmj}=\langle T_m,L_n^\varphi\varphi^{-1}U_j\rangle_\varphi=\frac{\pi}{n+1}\sum_{k=1}^n T_m(x_k)\widetilde{u}_j(x_k)=\alpha_{jn}^\varphi(T_m)$$

taking into account Relation (3.7). Hence, due to Corollary 3.3, $(M_n^\varphi\varphi^{-1}P_n)^*\varphi^{-1}T_m=M_n^\varphi T_m\longrightarrow T_m$ in $\mathbf{L}_\sigma^2$, and the lemma is completely proved.

We still need a consequence of the lifting principle Lemma 2.2.

Lemma 4.6. *If $\{A_n\}^o\in\mathcal{A}/\mathcal{J}$ and $W\{A_n\}:\mathbf{L}_\sigma^2\longrightarrow\mathbf{L}_\sigma^2$ as well as $\widetilde{W}\{A_n\}:\mathbf{L}_\sigma^2\longrightarrow\mathbf{L}_\sigma^2$ are invertible from the same side, then they are invertible from both sides.*

Proof. A closer look at the proof of Lemma 2.2 shows that also a one-sided version of that lemma holds: if $W\{A_n\}$, $\widetilde{W}\{A_n\}$ and $\{A_n\}^o$ are invertible from the same side, say from the right hand side, then the sequence $\{A_n\}$ is stable from that side in the sense that there is a sequence $\{B_n\}$ such that $A_nB_n=P_n+G_n$ with a sequence $\{G_n\}\in\mathcal{N}$. Since the A_n are matrices, this clearly implies the common stability of the sequence $\{A_n\}$. But then, due to Lemma 2.2, $W\{A_n\}$, $\widetilde{W}\{A_n\}$ and $\{A_n\}^o$ are two-sided invertible.

Corollary 4.7. *One has* $\{M_n^\varphi SP_n\}^o\{(M_n^\varphi SP_n)^*\}^o = \{P_n\}^o$ *and, hence,*

$$\{M_n^\varphi SP_n\}^o\{(M_n^\varphi SP_n)^*\}^o + \mathcal{J}_\tau^\varphi = \{P_n\}^o + \mathcal{J}_\tau^\varphi \quad \textit{for all} \quad \tau \in [-1,1],$$

whereas

$$\{(M_n^\varphi SP_n)^*\}^o\{M_n^\varphi SP_n\}^o + \mathcal{J}_\tau^\varphi \neq \{P_n\}^o + \mathcal{J}_\tau^\varphi \quad \text{for} \quad \tau = \pm 1\,. \tag{4.10}$$

Proof. By (3.14), (4.4), and (4.3),

$$\begin{aligned} M_n^\varphi SP_n(M_n^\varphi SP_n)^* &= \left(M_n^\varphi S\varphi SP_n - \frac{i}{2}M_n^\varphi SVP_nW_nP_1W_n\right)M_n^\varphi\varphi^{-1}P_n = \\ &= P_n + M_n^\varphi K_0 M_n^\varphi \varphi^{-1}P_n - \frac{i}{2}M_n^\varphi SVP_nW_nP_1W_nM_n^\varphi\varphi^{-1}P_n. \end{aligned}$$

It remains to show that the sequences

$$\{M_n^\varphi K_0 M_n^\varphi\varphi^{-1}P_n\} \quad \text{and} \quad \{W_nP_1W_nM_n^\varphi\varphi^{-1}P_n\} = \{W_nP_1M_n^\varphi\varphi^{-1}P_nW_n\}$$

belong to the ideal $\mathcal{J}$. This is a consequence of Lemma 4.5 and the relations

$$\|P_1u\|_\sigma = |\langle u, \widetilde{u}_0\rangle_\sigma| = \sqrt{2/\pi}\left|\int_{-1}^1 u(x)\,dx\right| \le \text{const}\ \|u\|_\varphi$$

and

$$\|K_0u\|_\sigma = \frac{1}{\sqrt{2}}\,|\langle u, \widetilde{u}_0\rangle_\sigma| \le \text{const}\ \|u\|_\varphi\,,$$

which imply the compactness of the operators $P_1 : \mathbf{L}^2_\varphi \longrightarrow \mathbf{L}^2_\sigma$ and $K_0 : \mathbf{L}^2_\varphi \longrightarrow \mathbf{L}^2_\sigma$.

Now assume that (4.10) is not true for $\tau = 1$, for example. Then it is also not true for $\tau = -1$ which can be seen as follows. Set $(Wf)(x) := f(-x)$. Then

$$WSW = -S\,, \quad WP_n = P_nW\,, \quad WM_n^\varphi = M_n^\varphi W\,, \quad \text{and} \quad \{P_nW\}^o\mathcal{J}_1^\varphi\{WP_n\}^o = \mathcal{J}_{-1}^\varphi$$

(observe that $W\widetilde{u}_k = (-1)^k\widetilde{u}_k$). Hence, applying W to

$$\{(M_n^\varphi SP_n)^*\}^o\{M_n^\varphi SP_n\}^o + \mathcal{J}_1^\varphi = \{P_n\}^o + \mathcal{J}_1^\varphi$$

yields

$$\{P_nW(M_n^\varphi SP_n)^*WP_n\}^o\{M_n^\varphi(-S)P_n\}^o + \mathcal{J}_{-1}^\varphi = \{P_n\}^o + \mathcal{J}_{-1}^\varphi\,.$$

Together with (4.5) and the local principle by Allan/Douglas, this leads to the invertibility of the coset $\{M_n^\varphi SP_n\}^o$ in contradiction to Lemma 4.6.

Thus, the coset $\{(M_n^\varphi SP_n)^*\}^o + \mathcal{J}_\tau^\varphi$ is an isometry in the local algebra $(\mathcal{A}^\varphi/\mathcal{J})/\mathcal{J}_\tau^\varphi$. Thanks to a result by Coburn [3], C^*-algebras generated by an isometry possess a nice description in terms of shift operators on the Hilbert space ℓ^2 of all square summable sequences of complex numbers. In particular, Coburn's theorem implies that the local algebra $(\mathcal{A}^\varphi/\mathcal{J})/\mathcal{J}_\tau^\varphi$ is *-isomorphic to the C^*-subalgebra of $\mathcal{L}(\ell^2)$ generated by the shift operator

$$\Sigma : \ell^2 \to \ell^2, \quad \{x_0, x_1, \ldots\} \mapsto \{0, x_0, x_1, \ldots\}$$

where the isomorphism sends $\{M_n^\varphi SP_n\}^o + \mathcal{J}_\tau^\varphi$ into

$$\Sigma^* : \ell^2 \to \ell^2, \quad \{x_0, x_1, \ldots\} \mapsto \{x_1, x_2, \ldots\}\,.$$

Applying the local principle of Allan and Douglas together with Lemma 2.2 and Lemma 3.6, we can summarize the considerations of this section.

Theorem 4.8. (*a*) *There is a *-isomorphism* η_ω *from* $\mathcal{A}^\omega/\mathcal{J}$ *onto a* C^**-algebra of bounded functions living on* $((-1,1)\times[0,1])\cup(\{\pm1\}\times\mathbb{T}_\tau)$ *in case* $\omega=\sigma$ *and on* $((-1,1)\times[0,1])\cup\{\pm1\}$ *in case* $\omega=\varphi$. *This isomorphism sends the coset* $\{M_n^\omega aP_n\}^o$ *into*

$$(x,\mu)\mapsto\begin{bmatrix} a(x+0)\mu+a(x)(1-\mu) & (a(x+0)-a(x))\sqrt{\mu(1-\mu)} \\ (a(x+0)-a(x))\sqrt{\mu(1-\mu)} & a(x+0)(1-\mu)+a(x)\mu \end{bmatrix}$$

and the coset $\{M_n^\omega SP_n\}^o$ *into*

$$(x,\mu)\mapsto\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

for $(x,\mu)\in(-1,1)\times[0,1]$. *Moreover, for* $(x,t)\in\{\pm1\}\times\mathbb{T}_\tau$,

$$\eta_\sigma\{M_n^\sigma(aI+bS)P_n\}^o(x,t)=a(x)+b(x)t$$

and, for $x=\pm1$,

$$\eta_\varphi\{M_n^\varphi(aI+bS)P_n\}^o(x)=a(x)I+b(x)\Sigma^*.$$

(*b*) *The sequence* $\{A_n\}\in\mathcal{A}^\omega$ *is stable if and only if the operators* $W\{A_n\},\widetilde{W}\{A_n\}:\mathbf{L}^2_\sigma\longrightarrow\mathbf{L}^2_\sigma$ *are invertible and if, in case* $\omega=\varphi$, *the operator* $\eta_\varphi\{A_n\}^o(x)$ *is invertible on* ℓ^2 *for* $x=\pm1$.

We remark that the invertibility of $W\{A_n\}$ already implies that $\det\eta_\omega\{A_n\}^o(x,\mu)\neq0$ for all $(x,\mu)\in(-1,1)\times[0,1]$ and that $\eta_\sigma\{A_n\}^o(\pm1,t)\neq0$ for all $t\in\mathbb{T}_{\pm1}$ (see Lemma 3.5).

For the stability of the sequence $\{A_n^\sigma\}=\{M_n^\sigma(aI+bS)P_n\}$, this theorem yields the invertibility of $W\{A_n^\sigma\}=aI+bS$ and of $\widetilde{W}\{A_n^\sigma\}=J_\sigma^{-1}(aJ_\sigma+ibV^*)$ as necessary and sufficient conditions. By Lemma 3.6, the invertibility of $\widetilde{W}\{A_n^\sigma\}$ is a consequence of the invertibility of $W\{A_n^\sigma\}$. Similarly, the sequence $\{A_n^\varphi\}=\{M_n^\varphi(aI+bS)P_n\}$ proves to be stable if and only if the operators $W\{A_n^\varphi\}=aI+bS$ and $\widetilde{W}\{A_n^\varphi\}=aI-bS$ on $\mathbf{L}^2_\sigma$ and the operators $a(\pm1)I+b(\pm1)\Sigma^*$ on ℓ^2 are invertible. It is easy to see that the invertibility of the latter operators is equivalent to the condition $a(\pm1)+b(\pm1)z\neq0$ for all $z\in\mathbb{C}$ with $|z|\le1$ which, on its hand, is already a consequence of the invertibility of $aI\pm bS$. This proves Theorem 2.1.

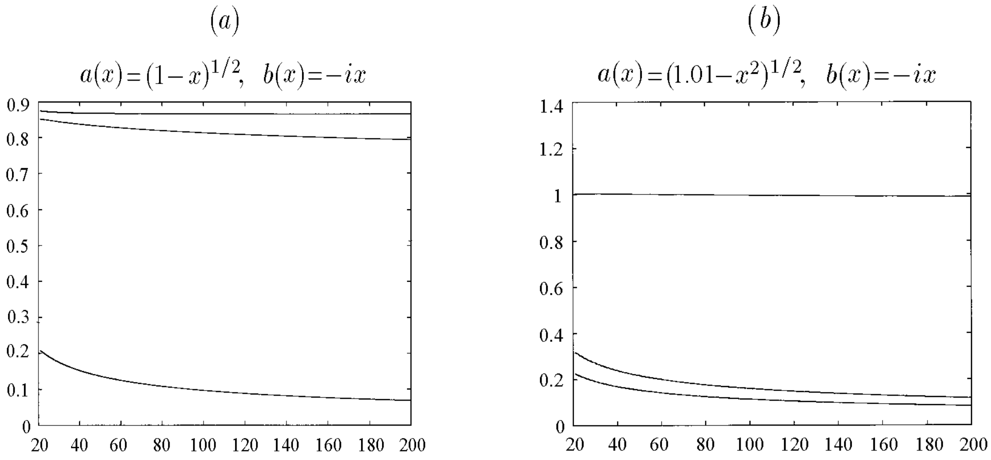
The assertion (b) of Theorem 4.8 can be easily translated into the case of a system of CSIE's

$$\sum_{k=1}^m\left[a_{jk}(x)u_k(x)+\frac{b_{jk}(x)}{\pi i}\int_{-1}^1\frac{u_k(y)}{y-x}\,dy\right]=f_j(x)\,,\quad -1<x<1\,,\ j=1,\,\ldots,\,m\,, \tag{4.11}$$

with piecewise continuous coefficients a_{jk} and b_{jk}. Indeed, denote by $\left\{A_n^{j,k}\right\}$ the operator sequence of the collocation method for (1.1) with a_{jk} and b_{jk} instead of a and b, respectively. Then the collocation method for (4.11) is stable in $(\mathbf{L}^2_\sigma)^m$ if and only if $\left[W\left\{A_n^{j,k}\right\}\right]_{j,k=1}^m$ and $\left[\widetilde{W}\left\{A_n^{j,k}\right\}\right]_{j,k=1}^m$ are invertible and if, in case $\omega=\varphi$, $\left[\eta_\varphi\left\{A_n^{j,k}\right\}^o(x)\right]_{j,k=1}^m$ is invertible for $x=\pm1$.

5. Behaviour of the smallest singular values

The singular values of a matrix A are the non-negative square roots of the eigenvalues of A^*A. The singular values of a matrix $A_n\in\mathbb{C}^{n\times n}$ will be denoted by $0\le\sigma_1^{(n)}\le\ldots\le\sigma_n^{(n)}$, counted with respect to their multiplicity.



The smallest three singular values of $A_n = M_n^\varphi(aI + bS)P_n$.

If $\{A_n\} \in \mathcal{F}$ is a stable sequence of matrices, then there is a positive constant C such that the smallest singular value $\sigma_1^{(n)}$ of A_n (hence, every singular value of A_n) is greater than C for all n, and conversely. Thus, if $\{A_n\}$ is non-stable, then there is a subsequence of the sequence $(\sigma_1^{(n)})$ which tends to zero. Figures (a) and (b) illustrate this behaviour for the non-stable sequences $\{A_n\}$ with $A_n = M_n^\varphi(aI + bS)P_n$, where $a(x) = \sqrt{1-x}$, $b(x) = -ix$ and $a(x) = \sqrt{1.01 - x^2}$, $b(x) = -ix$, respectively. In both cases we observe that not only a subsequence of $(\sigma_1^{(n)})$ but the sequence itself tends to zero. Moreover, in Figure (b), also the sequence $(\sigma_2^{(n)})$ of the second singular values goes to zero, whereas all other singular values are uniformly bounded from below by a positive constant. It is the goal of the present section to explain this effect and to derive a formula for the number of the singular values of A_n which tend to zero. Here we restrict ourselves to the case $\omega = \varphi$, although analogous considerations are possible for $\omega = \sigma$.

The desired results are closely related with a Fredholm theory for approximation sequences which has been developed in [20] and [18]. For the reader's convenience, we start with recalling some definitions and results from [20] and [18].

Fractal algebras. This class of subalgebras of $\mathcal{F}$ has been introduced and studied in [19, 17]. We will see in a moment that the algebra $\mathcal{A}^\varphi$ is fractal, and that the property of fractality is responsible for the fact that the complete sequence of the smallest singular values of a non-stable sequence $\{A_n\} \in \mathcal{A}^\varphi$ tends to zero and not only one of its proper subsequences.

Given a strongly monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$, let $\mathcal{F}_\eta$ refer to the C^*-algebra of all bounded sequences $\{A_n\}$ with $A_n \in \mathbb{C}^{\eta(n)\times\eta(n)}$, and write $\mathcal{N}_\eta$ for the ideal of all sequences $\{A_n\} \in \mathcal{F}_\eta$ which tend to zero in the norm. Further, let R_η stand for the restriction mapping $R_\eta : \mathcal{F} \to \mathcal{F}_\eta$, $\{A_n\} \mapsto \{A_{\eta(n)}\}$. This mapping is a *-homomorphism from $\mathcal{F}$ onto $\mathcal{F}_\eta$ which moreover maps $\mathcal{N}$ onto $\mathcal{N}_\eta$. Given a C^*-subalgebra $\mathcal{A}$ of $\mathcal{F}$, let $\mathcal{A}_\eta$ denote the image of $\mathcal{A}$ under R_η which is a C^*-algebra again.

Definition 5.1. *Let $\mathcal{A}$ be a C^*-subalgebra of the algebra $\mathcal{F}$.*

(a) A *-homomorphism $W : \mathcal{A} \to \mathcal{B}$ of $\mathcal{A}$ into a C^*-algebra $\mathcal{B}$ is fractal *if, for every strongly monotonically increasing sequence η, there is a *-homomorphism $W_\eta : \mathcal{A}_\eta \to \mathcal{B}$ such that $W = W_\eta R_\eta$.*

(b) The algebra $\mathcal{A}$ is fractal *if the canonical homomorphism $\pi : \mathcal{A} \to \mathcal{A}/(\mathcal{A}\cap\mathcal{N})$ is fractal.*

Thus, given a subsequence $\{A_{\eta(n)}\}$ of a sequence $\{A_n\}$ which belongs to a fractal algebra $\mathcal{A}$, it is possible to reconstruct the original sequence $\{A_n\}$ from this subsequence modulo sequences in $\mathcal{A}\cap\mathcal{N}$. This assumption is very natural for sequences arising from discretization procedures. On the other hand, the algebra $\mathcal{F}$ of all bounded sequences fails to be fractal. The following theorem is shown in [17] and will easily imply the fractality of the algebra $\mathcal{A}^\varphi$.

Theorem 5.2. *Let $\mathcal{A}$ be a unital C^*-subalgebra of $\mathcal{F}$. The algebra $\mathcal{A}$ is fractal if and only if there exists a family $\{W_t\}_{t\in T}$ of unital and fractal *-homomorphisms W_t from $\mathcal{A}$ into unital C^*-algebras $\mathcal{B}_t$ such that the following equivalence holds for every sequence $\{A_n\}\in\mathcal{A}$: The coset $\{A_n\}+\mathcal{A}\cap\mathcal{N}$ is invertible in $\mathcal{A}/(\mathcal{A}\cap\mathcal{N})$ if and only if $W_t\{A_n\}$ is invertible in $\mathcal{B}_t$ for every $t\in T$.*

To make the proof of the fractality of the algebra $\mathcal{A}^\varphi$ more transparent, we introduce a few new notations and rewrite Theorem 4.8 as follows. Set $T:=\{1,2,3,4\}$ and define *-homomorphisms $W_1, W_2:\mathcal{F}^W\to\mathcal{L}(\mathbf{L}^2_\sigma)$ and $W_3, W_4:\mathcal{F}^W\to\mathcal{L}(l^2)$ by $W_1:=W$, $W_2:=\widetilde{W}$ and

$$W_3\{A_n\}:=\eta_\varphi\{A_n\}^o(1),\quad W_4\{A_n\}:=\eta_\varphi\{A_n\}^o(-1).$$

Theorem 4.8'. *(a) A sequence $\{A_n\}\in\mathcal{A}^\varphi$ is stable if and only if the operators $W_t\{A_n\}$ are invertible for all $t\in T$.*

(b) The mapping

$$\mathrm{smb}:\mathcal{A}^\varphi\to\mathcal{L}(\mathbf{L}^2_\sigma)\times\mathcal{L}(\mathbf{L}^2_\sigma)\times\mathcal{L}(l^2)\times\mathcal{L}(l^2),$$

$$\{A_n\}\mapsto(W_1\{A_n\},\,W_2\{A_n\},\,W_3\{A_n\},\,W_4\{A_n\})$$

is a *-homomorphism with kernel $\mathcal{N}$.

The first assertion is just a reformulation of Theorem 4.8, and the second one is a simple consequence of the fact that every *-homomorphism between C^*-algebras which preserves spectra also preserves norms.

Corollary 5.3. *The algebra $\mathcal{A}^\varphi$ is fractal.*

Proof. By Theorem 5.2, we have to prove that all homomorphisms W_t are fractal. For W_1 and W_2, the fractality is evident: these homomorphisms act as strong limits, and the strong limit of a subsequence of $\{A_n\}$ coincides with the strong limit of $\{A_n\}$ itself. Concerning W_3 and W_4, a closer look at the proof of Corollary 4.7 shows that the assertion of that corollary remains valid for every infinite subsequence of $\{M_n^\varphi SP_n\}$ in place of the sequence $\{M_n^\varphi SP_n\}$ itself. Thus, Coburn's theorem again applies, yielding the fractality of W_3 and W_4.

Fredholm sequences. Let $\mathcal{J}(\mathcal{F})$ stand for the smallest closed subset of $\mathcal{F}$ which contains all sequences $\{K_n\}$ for which $\sup\dim\operatorname{Im}K_n$ is finite. The set $\mathcal{J}(\mathcal{F})$ is a closed two-sided ideal of $\mathcal{F}$ which contains the ideal $\mathcal{N}$ of the zero sequences. A sequence $\{A_n\}\in\mathcal{F}$ is called a *Fredholm sequence* if it is invertible modulo the ideal $\mathcal{J}(\mathcal{F})$. If $\{A_n\}$ is a Fredholm sequence then there is a number k such that $\liminf_{n\to\infty}\sigma_{k+1}^{(n)}>0$ (see [18, Theorem 2]). The smallest number k with this property is called the *α-number* of the sequence $\{A_n\}$ and will be denoted by $\alpha\{A_n\}$. This number plays the same role in the Fredholm theory of approximation sequences as the number $\dim\operatorname{Ker}A$ plays in the common Fredholm theory for operators A on a Hilbert space.

The remainder of this section is devoted to the proof of the following result which characterizes the Fredholm sequences in $\mathcal{A}^\varphi$.

Theorem 5.4. *(a) A sequence $\{A_n\} \in \mathcal{A}^\varphi$ is Fredholm if and only if the operators $W_t\{A_n\}$ are Fredholm operators for every $t \in T$.*

(b) If $\{A_n\} \in \mathcal{A}^\varphi$ is a Fredholm sequence, then

$$\alpha\{A_n\} = \dim \operatorname{Ker} W_1\{A_n\} + \dim \operatorname{Ker} W_2\{A_n\} + \dim \operatorname{Ker} W_3\{A_n\} + \dim \operatorname{Ker} W_4\{A_n\}.$$

(c) If $\{A_n\} \in \mathcal{A}^\varphi$ is Fredholm and $k = \alpha\{A_n\} > 0$, then $\lim_{n\to\infty} \sigma_k^{(n)} = 0$.

Fredholm inverse closed subalgebras. Let $\mathcal{A}$ be a unital and fractal C^*-subalgebra of $\mathcal{F}$ which contains the ideal $\mathcal{N}$. A sequence $\{K_n\}$ in $\mathcal{A}$ is said to be of *central rank one* if, for every sequence $\{A_n\} \in \mathcal{A}$, there is a sequence $\{\mu_n\} \in c$ (= the set of all convergent sequences of complex numbers) such that

$$K_n A_n K_n = \mu_n K_n.$$

The smallest closed two-sided ideal of $\mathcal{A}$ which contains all sequences of central rank one will be denoted by $\mathcal{J}(\mathcal{A})$. The algebra $\mathcal{A}$ is called *Fredholm inverse closed* in $\mathcal{F}$ if $\mathcal{J}(\mathcal{A}) = \mathcal{A} \cap \mathcal{J}(\mathcal{F})$.

Sequences of essential rank one. Let $\mathcal{A}$ be as before. A central rank one sequence of $\mathcal{A}$ is said to be of *essential rank one* if it does not belong to the ideal $\mathcal{N}$. For every essential rank one sequence $\{K_n\}$, let $J\{K_n\}$ refer to the smallest closed ideal of $\mathcal{A}$ which contains the sequence $\{K_n\}$ and the ideal $\mathcal{N}$. In [18] it is shown that, if $\{K_n\}$ and $\{L_n\}$ are sequences of essential rank one in $\mathcal{A}$, then either $J\{K_n\} = J\{L_n\}$ or $J\{K_n\} \cap J\{L_n\} = \mathcal{N}$. Calling $\{K_n\}$ and $\{L_n\}$ equivalent in the first case we get a splitting of the sequences of essential rank one into equivalence classes, which we denote by S. Further, with every $s \in S$, there is associated a unique irreducible representation W_s of $\mathcal{A}$ into the algebra $\mathcal{L}(H_s)$ for some Hilbert space H_s such that the ideal $J\{K_n\}$ is mapped onto the ideal $K(H_s)$ of the compact operators on H_s and that the kernel of the mapping $W_s : J\{K_n\} \to K(H_s)$ is $\mathcal{N}$. The main result of [18] reads as follows:

Theorem 5.5. *Let $\mathcal{A}$ be a unital, fractal and Fredholm inverse closed C^*-subalgebra of $\mathcal{F}$ which contains the ideal $\mathcal{N}$.*

(a) If $\{A_n\} \in \mathcal{A}$ is a Fredholm sequence, then the operators $W_s\{A_n\}$ are Fredholm operators for every $s \in S$, and $\alpha\{A_n\} = \sum_{s\in S} \dim \operatorname{Ker} W_s\{A_n\}$.

(b) If $\{A_n\} \in \mathcal{A}$ is Fredholm and $k = \alpha\{A_n\} > 0$, then $\lim_{n\to\infty} \sigma_k^{(n)} = 0$.

(c) If the family $(W_s)_{s\in S}$ is sufficient for the stability of sequences in $\mathcal{A}$ (in the sense that the invertibility of all operators $W_s\{A_n\}$ implies the stability of $\{A_n\}$) and if all operators $W_s\{A_n\}$ are Fredholm for a sequence $\{A_n\} \in \mathcal{A}$, then this sequence is Fredholm.

We know already that $\mathcal{A}^\varphi$ is a fractal algebra. Thus, once we have shown that this algebra is Fredholm inverse closed and once we have identified the set S with $T = \{1, 2, 3, 4\}$ as well as the representations W_s with the corresponding homomorphisms W_t figuring in Theorem 5.2, then Theorem 5.4 is proved.

Another type of "compact"sequences. Let again $\mathcal{A}$ refer to a unital C^*-subalgebra of $\mathcal{F}$. Besides the ideal $\mathcal{J}(\mathcal{A})$ we consider a further ideal, $\mathcal{K}(\mathcal{A})$, which is the smallest closed two-sided ideal of $\mathcal{A}$ containing all sequences $\{K_n\} \in \mathcal{A}$ with $\dim \operatorname{Im} K_n \le 1$ for all n.

Proposition 5.6. *Let $\mathcal{A}$ be a unital and fractal C^*-subalgebra of $\mathcal{F}$ which contains $\mathcal{N}$. Then, $\mathcal{J}(\mathcal{A}) = \mathcal{K}(\mathcal{A})$.*

Proof. If $\{K_n\}$ is a central rank one sequence in $\mathcal{A}$ then, since $\mathcal{N} \subseteq \mathcal{A}$, every matrix K_n has rank one. Thus, $\{K_n\}$ belongs to $\mathcal{K}(\mathcal{A})$.

For the reverse inclusion, first observe that, under the made assumptions, the center of $\mathcal{A}$ consists exactly of all sequences of the form $\{\alpha_n P_n\}$ where $\{\alpha_n\}$ is in c, the set of all convergent sequences. Now let $\{K_n\} \in \mathcal{A}$ be a sequence with $\dim \operatorname{Im} K_n \leq 1$ for all n. The fractality of $\mathcal{A}$ further implies the existence of the limit $\alpha := \lim \|K_n\|$ (see [17, Theorem 4]). If $\alpha = 0$, then $\{K_n\}$ is a zero sequence, hence in $\mathcal{N} \subseteq \mathcal{K}(\mathcal{A})$.

In case $\alpha \neq 0$ we are going to show that $\{K_n\}$ is a central rank one sequence. Assume $\{K_n\}$ is not of central rank one. Then there are a sequence $\{A_n\} \in \mathcal{A}$ and a non-convergent sequence $(\alpha_n) \in l^\infty$ such that

$$K_n A_n K_n = \alpha_n K_n \quad \text{for all } n.$$

Choose two partial limits $\beta \neq \gamma$ of the sequence (α_n) as well as two subsequences μ and η of the the positive integers such that

$$\alpha_{\mu(n)} \to \beta \quad \text{and} \quad \alpha_{\eta(n)} \to \gamma \quad \text{as } n \to \infty.$$

Then both sequences $\{\alpha_{\mu(n)} K_{\mu(n)} - \beta K_{\mu(n)}\}$ and $\{\alpha_{\eta(n)} K_{\eta(n)} - \gamma K_{\eta(n)}\}$ tend to zero. Hence, again due to the fractality of $\mathcal{A}$ (see [17, Theorem 1]), both sequences $\{\alpha_n K_n - \beta K_n\}$ and $\{\alpha_n K_n - \gamma K_n\}$ are zero sequences. But then, also their difference $(\beta - \gamma)\{K_n\}$ goes to zero. Since $\|K_n\| \to \alpha \neq 0$, this implies $\beta = \gamma$ in contradiction to the choice of β and γ.

Identification of the ideals $\mathcal{J}(\mathcal{A}^\varphi) = \mathcal{K}(\mathcal{A}^\varphi)$. Our next objective is to characterize the image of the ideal $\mathcal{J}(\mathcal{A}^\varphi)$ under the mapping smb introduced in Theorem 4.8'.

Theorem 5.7. *The homomorphism* smb *maps* $\mathcal{J}(\mathcal{A}^\varphi)$ *onto* $K(\mathbf{L}^2_\sigma) \times K(\mathbf{L}^2_\sigma) \times K(l^2) \times K(l^2)$.

Proof. It is shown in [18, Theorem 3] that every irreducible representation of a C^*-algebra $\mathcal{A}$ maps every central rank one element of $\mathcal{A}$ onto a compact operator (an element k of a C^*-algebra $\mathcal{A}$ is of central rank one if, for every $a \in \mathcal{A}$, there is an element μ in the center of $\mathcal{A}$ such that $kak = \mu k$). In our setting, the homomorphisms W_t, $1 \leq t \leq 4$ are irreducible since, in any case, the ideal of the compact operators belongs to the image of $\mathcal{A}^\varphi$ under W_t. Thus,

$$\operatorname{smb}(\mathcal{J}(\mathcal{A}^\varphi)) \subseteq K(\mathbf{L}^2_\sigma) \times K(\mathbf{L}^2_\sigma) \times K(l^2) \times K(l^2).$$

For the reverse inclusion first recall that, by definition, the set $\mathcal{J}$ of all sequences $\{P_n K P_n + W_n L W_n + C_n\}$ with K, L compact and $\{C_n\} \in \mathcal{N}$ is contained in $\mathcal{A}^\varphi$. Since every compact operator can approximated as closely as desired by an operator with finite dimensional range, we have $\mathcal{J} \subseteq \mathcal{K}(\mathcal{A}^\varphi)$ and thus, by Proposition 5.6, $\mathcal{J} \subseteq \mathcal{J}(\mathcal{A}^\varphi)$. Moreover, it is easy to check for the sequence $\{K_n\} := \{P_n K P_n + W_n L W_n + C_n\}$ that

$$W_1\{K_n\} = K \quad \text{and} \quad W_2\{K_n\} = L,$$

and it is immediate from the definition of W_3 and W_4 that $W_3\{K_n\} = W_4\{K_n\} = 0$. Hence, $K(\mathbf{L}^2_\sigma) \times K(\mathbf{L}^2_\sigma) \times \{0\} \times \{0\}$ lies in $\operatorname{smb}(\mathcal{J}(\mathcal{A}^\varphi))$. So it remains to show that $\operatorname{smb}(\mathcal{J}(\mathcal{A}^\varphi))$ contains all quadrupels of the form $(0, 0, K, 0)$ and $(0, 0, 0, K)$ with K a compact operator on l^2.

It is well known and easy to check that the smallest closed C^*-subalgebra of $\mathcal{L}(l^2)$ which contains the shift operator Σ also contains all compact operators and that, in particular, $K(l^2)$

is the smallest closed ideal of that algebra which contains the projection $\Pi := I - \Sigma\Sigma^*$ acting by

$$\Pi : l^2 \to l^2, \quad \{x_0, x_1, \ldots\} \mapsto \{x_0, 0, 0, \ldots\}.$$

Because of $W_t\{M_n^\varphi SP_n\} = \Sigma^*$ for $t = 3$ and $t = 4$, it is consequently sufficient to prove that the quadrupels $(0, 0, \Pi, 0)$ and $(0, 0, 0, \Pi)$ belong to $\mathrm{smb}(\mathcal{J}(\mathcal{A}^\varphi))$.

For this goal, we consider the sequences $\{A_n\}$ and $\{B_n\}$ with

$$A_n := P_n - (M_n^\varphi SP_n)^* M_n^\varphi SP_n \quad \text{and} \quad B_n := M_n^\varphi VP_n - (M_n^\varphi SP_n)^* M_n^\varphi SVP_n$$

where, as above, $V = \psi I - i\varphi S$ and $\psi(x) = x$. For the sequence $\{A_n\}$ we have $W_1\{A_n\} = I - \varphi S\varphi^{-1}S$ (Section 2), and this operator is 0 as we have already mentioned several times. Similarly, $W_2\{A_n\} = I - (-\varphi S\varphi^{-1})(-S) = 0$ due to Theorem 3.4. It is further immediate from the definitions that $W_3\{A_n\} = W_4\{A_n\} = \Pi$; thus, $\mathrm{smb}\,\{A_n\} = (0, 0, \Pi, \Pi)$.

Concerning the sequence $\{B_n\}$, we will first show that it is indeed an element of the algebra $\mathcal{A}^\varphi$. To see this, write $SV = S(\psi I - i\varphi S)$ as

$$SV = \psi S - i\varphi I + (S\psi I - \psi S) - iK_0 \tag{5.12}$$

where the rank one operator K_0 is given by (4.3). Thus, SV is the sum of a singular integral operator with continuous coefficients and of a compact operator, which maps into the convergence manifold of the M_n^φ. Hence, $\{M_n^\varphi SVP_n\} \in \mathcal{A}^\varphi$ and $\{B_n\} \in \mathcal{A}^\varphi$. To compute the operators $W_t\{B_n\}$, we recall from Section 2 that

$$W_1\{M_n^\varphi VP_n\} = W_1\{M_n^\varphi(\psi I - i\varphi S)P_n\} = \psi I - i\varphi S = V, \quad W_2\{M_n^\varphi VP_n\} = \psi I + i\varphi S,$$

$$W_3\{M_n^\varphi VP_n\} = W_3\{M_n^\varphi(\psi(1)I - i\varphi(1)S)P_n\} = I \quad \text{and} \quad W_4\{M_n^\varphi VP_n\} = -I$$

and that

$$W_1\{M_n^\varphi SP_n\}^* = -W_2\{M_n^\varphi SP_n\}^* = \varphi S\varphi^{-1}I \quad \text{and} \quad W_3\{M_n^\varphi SP_n\}^* = W_4\{M_n^\varphi SP_n\}^* = \Sigma.$$

For the operators $W_t\{M_n^\varphi SVP_n\}$ we make use of identity (5.12) which together with the results of Section 2 yields

$$W_1\{M_n^\varphi SVP_n\} = SV, \quad W_2\{M_n^\varphi SVP_n\} = -i\varphi I - \psi S,$$

$$W_3\{M_n^\varphi SVP_n\} = W_3\{M_n^\varphi(\psi(1)S - i\varphi(1)I)P_n\} = \Sigma^* \quad \text{and} \quad W_4\{M_n^\varphi SVP_n\} = -\Sigma^*.$$

Puzzling these pieces together we find

$$W_1\{B_n\} = V - \varphi S\varphi^{-1}SV = (I - \varphi S\varphi^{-1}S)V = 0,$$

and

$$W_2\{B_n\} = (\psi I + i\varphi S) + \varphi S\varphi^{-1}(-\psi S - i\varphi I) = \varphi S\varphi^{-1}(S\psi I - \psi S)$$

which is also 0 since the range of the commutator $S\psi I - \psi S$ consists of constant functions only and since the operator $S\varphi^{-1}$ annihilates every constant function. Finally,

$$W_3\{B_n\} = I - \Sigma\Sigma^* = \Pi \quad \text{and} \quad W_4\{B_n\} = -I - \Sigma(-\Sigma^*) = -\Pi,$$

whence $\mathrm{smb}\,\{B_n\} = (0, 0, \Pi, -\Pi)$.

Thus, the homomorphism smb maps the sequences $\frac{1}{2}\{A_n + B_n\}$ and $\frac{1}{2}\{A_n - B_n\}$ onto the quadrupels $(0, 0, \Pi, 0)$ and $(0, 0, 0, \Pi)$, respectively. We show that these sequences are essential rank one sequences in $\mathcal{A}^\varphi$. For this goal, we determine the matrix representation of $A_n = P_n - (M_n^\varphi S P_n)^* M_n^\varphi S P_n$ with respect to the basis functions $\tilde{u}_k$, $k = 0, \ldots, n-1$. For $0 \le k, m \le n-1$ we have

$$\begin{aligned} \tau_{km} &:= \langle (M_n^\varphi S P_n)^* M_n^\varphi S P_n \tilde{u}_m, \tilde{u}_k \rangle_\sigma = \\ &= \langle M_n^\varphi S P_n \tilde{u}_m, M_n^\varphi S P_n \tilde{u}_k \rangle_\sigma = \langle M_n^\varphi S \tilde{u}_m, M_n^\varphi S \tilde{u}_k \rangle_\sigma \end{aligned}$$

and, since $S\tilde{u}_m = S\varphi U_m = iT_{m+1}$, we obtain

$$\begin{aligned} \tau_{km} &= \langle M_n^\varphi T_{m+1}, M_n^\varphi T_{k+1} \rangle_\sigma = \langle L_n^\varphi \varphi^{-1} T_{m+1}, L_n^\varphi \varphi^{-1} T_{k+1} \rangle_\varphi = \\ &= \frac{\pi}{n+1} \sum_{l=1}^{n} T_{m+1}(x_{ln}^\varphi) T_{k+1}(x_{ln}^\varphi) = \\ &= \frac{2}{n+1} \sum_{l=1}^{n} \cos\frac{(m+1)l\pi}{n+1} \cos\frac{(k+1)l\pi}{n+1} = \\ &= \frac{1}{n+1} \sum_{l=1}^{n} \left(\cos\frac{(m-k)l\pi}{n+1} + \cos\frac{(m+k+2)l\pi}{n+1} \right). \end{aligned}$$

Short calculations using the well known identity

$$\sum_{l=1}^{n} \cos lx = \frac{\sin\left(n + \frac{1}{2}\right)x}{2\sin\frac{x}{2}} - \frac{1}{2}$$

yield in case $k = m$

$$\tau_{mm} = \frac{n-1}{n+1},$$

whereas in case $k \neq m$

$$\tau_{km} = \begin{cases} 0 & \text{if } m+k \text{ is odd} \\ -\dfrac{2}{n+1} & \text{if } m+k \text{ is even.} \end{cases}$$

Summarizing we find that $A_n =: \dfrac{2}{n+1} [\varepsilon_{km}]_{k,m=0}^{n-1}$ with

$$\varepsilon_{km} = \begin{cases} 0 & \text{if } m+k \text{ is odd} \\ 1 & \text{if } m+k \text{ is even,} \end{cases}$$

and analogously one gets that the matrix representation of B_n with respect to the same basis is $B_n =: \dfrac{2}{n+1} [\varepsilon_{k,m+1}]_{k,m=0}^{n-1}$. It is evident now that the sequences $\frac{1}{2}\{A_n + B_n\}$ and $\frac{1}{2}\{A_n - B_n\}$ consist of matrices with rank one only. Thus, by Proposition 5.6, these sequences are of essential rank one, and this observation finishes the proof of the inclusion

$$K(\mathbf{L}_\sigma^2) \times K(\mathbf{L}_\sigma^2) \times K(l^2) \times K(l^2) \subseteq \mathrm{smb}(\mathcal{J}(\mathcal{A}^\varphi))$$

and of Theorem 5.7.

Fredholm inverse closedness of $\mathcal{A}^\varphi$. To finish also the proof of Theorem 5.4, we have finally to show that the ideals $\mathcal{J}(\mathcal{A}^\varphi) = \mathcal{K}(\mathcal{A}^\varphi)$ and $\mathcal{J}(\mathcal{F}) \cap \mathcal{A}^\varphi$ of $\mathcal{A}^\varphi$ coincide. This equality is a simple consequence of the following result which, on its hand, is a generalization of [18, Theorem 3].

Theorem 5.8. *Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{F}$ and let $\{J_n\}$ be a sequence in $\mathcal{J}(\mathcal{F}) \cap \mathcal{A}$. Then, for every irreducible representation $W : \mathcal{A} \to \mathcal{L}(K)$ of $\mathcal{A}$, the operator $W\{J_n\}$ is compact.*

Proof. The proof is based on [15, Prop. 4.1.8] which states that, under the above assumptions, there exist an irreducible representation $\pi : \mathcal{F} \to \mathcal{L}(H)$ of $\mathcal{F}$ with a certain Hilbert space H, a subspace H_1 of H and an isometry U from H_1 onto K such that H_1 is an invariant subspace for $\pi\{J_n\}$ and

$$W\{J_n\} = U\pi\{J_n\}|_{H_1} U^*.$$

From [18, Theorem 3] we know that $\pi\{J_n\}$ is a compact operator on H. Since H_1 is invariant for $\pi\{J_n\}$, this moreover implies that $\pi\{J_n\}|_{H_1}$ is a compact operator on H_1. Thus, $W\{J_n\}$ is compact on K.

The Figures (*a*) and (*b*) revisited. Having Theorem 5.4 at our disposal, it is easy to explain the behaviour of the smallest singular values in Figures (a) and (b). In case $A = aI + bS$, $a(x) = \sqrt{1-x}$, $b(x) = -ix$, we have for $A_n = M_n^\varphi A P_n$

$$\dim \operatorname{Ker} W_1\{A_n\} + \dim \operatorname{Ker} W_2\{A_n\} + \dim \operatorname{Ker} W_3\{A_n\} + \dim \operatorname{Ker} W_4\{A_n\} =$$

$$= 0 + 0 + 1 + 0 = 1$$

whereas the same quantity is $0 + 0 + 1 + 1 = 2$ in case $a(x) = \sqrt{1.01 - x^2}$, $b(x) = -ix$. Thus, in Figure (a) the lowest singular value tends to zero and the $\sigma_2^{(n)}$ remain bounded away from zero by a positive constant for all n, whereas in Figure (b) both $\lim \sigma_1^{(n)} = 0$ and $\lim \sigma_2^{(n)} = 0$.

6. Appendix: proof of lemma 4.4

Lemma 6.1. *The coset $\{M_n^\sigma S P_n\}^o$ is a unitary element of $\mathcal{A}^\sigma/\mathcal{J}$.*

Proof. We use (3.15) and (4.3) and get

$$\begin{aligned} M_n^\sigma S P_n (M_n^\sigma S P_n)^* &= \frac{1}{2} M_n^\sigma S \varphi S \varphi^{-1} L_n^\sigma (P_{n-1} + P_n) = \\ &= \frac{1}{2} M_n^\sigma L_n^\sigma (P_{n-1} + P_n) + \frac{1}{2} M_n^\sigma K_0 \varphi^{-1} L_n^\sigma (P_{n-1} + P_n) = \\ &= \frac{1}{2}(P_{n-1} + P_n) + \frac{1}{2} M_n^\sigma K_0 \varphi^{-1} L_n^\sigma (P_{n-1} + P_n)\,. \end{aligned}$$

Now, from $K_0 \varphi^{-1} u = -\frac{1}{\sqrt{2\pi}} \langle u, U_0 \rangle_\sigma$,

$$\begin{aligned} \|L_n^\sigma P_n f\|_\sigma^2 &= \frac{\pi}{n} \sum_{k=1}^n \left[1 - (x_{kn}^\sigma)^2\right] \left|(\varphi^{-1} P_n f)(x_{kn}^\sigma)\right|^2 \le \\ &\le \text{const} \int_{-1}^1 \varphi(x) \left|(\varphi^{-1} P_n f)(x)\right|^2 dx = \text{const}\ \|P_n f\|_\sigma^2 \end{aligned}$$

(see Lemma 3.1), and $P_{n-1} = P_n - W_n P_1 W_n$, it follows $\{M_n^\sigma S P_n\}^o\{(M_n^\sigma S P_n)^*\}^o = \{P_n\}^o$. On the other hand we have

$$(M_n^\sigma S P_n)^* M_n^\sigma S P_n = \varphi S \varphi^{-1} L_n^\sigma M_n^\sigma S P_n - \frac{1}{2}\varphi S \varphi^{-1} L_n^\sigma W_n P_1 W_n M_n^\sigma S P_n$$

and $\varphi S \varphi^{-1} L_n^\sigma M_n^\sigma S P_n = \varphi S \varphi^{-1} S P_n = P_n$ due to (3.6) and (3.16).

For $n \geq 1$, let $Q_n : \ell^2 \longrightarrow \ell^2$ denote the projection $Q_n\xi = \{\xi_0, \ldots, \xi_{n-1}, 0, 0, \ldots\}$, and define $E_n^\pm : \operatorname{im} P_n \longrightarrow \operatorname{im+} Q_n$ by

$$E_n^+ u = \left\{ \sqrt{\frac{\pi}{n}} u(x_{1n}^\sigma), \ldots, \sqrt{\frac{\pi}{n}} u(x_{nn}^\sigma), 0, \ldots \right\}$$

and

$$E_n^- u = \left\{ \sqrt{\frac{\pi}{n}} u(x_{nn}^\sigma), \ldots, \sqrt{\frac{\pi}{n}} u(x_{1n}^\sigma), 0, \ldots \right\}.$$

Then, for $\xi \in \operatorname{im} Q_n$,

$$(E_n^+)^{-1}\xi = \sum_{k=1}^n \sqrt{\frac{n}{\pi}} \xi_{k-1} \widetilde{\ell}_{kn}^\sigma =: E_{-n}^+ \xi \quad \text{and} \quad (E_n^-)^{-1}\xi = \sum_{k=1}^n \sqrt{\frac{n}{\pi}} \xi_{n-k} \widetilde{\ell}_{kn}^\sigma =: E_{-n}^- \xi\,,$$

where

$$\widetilde{\ell}_{kn}^\sigma(x) = \frac{\varphi(x) T_n(x)}{\varphi(x_{kn}^\sigma)(x - x_{kn}^\sigma) T_n'(x_{kn}^\sigma)}\,.$$

Remark that, for $n \geq 1$,

$$T_n'(x) = n U_{n-1}(x) \quad \text{and} \quad T_n'(x_{kn}^\sigma) = \sqrt{\frac{2}{\pi}} \frac{n(-1)^{k+1}}{\varphi(x_{kn}^\sigma)}\,.$$

In view of Lemma 3.1 and estimate (3.1), the sequences $\{E_n^\pm\}$ and $\{E_{-n}^\pm\}$ are uniformly bounded, i. e. there are constants c_1 and c_2 such that

$$\frac{\pi}{n} \sum_{k=1}^n |u(x_{kn}^\sigma)|^2 \leq c_1 \|u\|_\sigma^2 \quad \text{for all} \quad u \in \operatorname{im} P_n \tag{6.1}$$

and

$$\left\| \sqrt{\frac{n}{\pi}} \sum_{k=1}^n \xi_{k-1} \widetilde{\ell}_{kn}^\sigma \right\|_\sigma^2 \leq c_2 \sum_{k=1}^n |\xi_{k-1}|^2 \quad \text{for all} \quad \xi \in \ell^2\,. \tag{6.2}$$

Lemma 6.2. *The sequences $\{E_n^\pm M_n^\sigma S P_n E_{-n}^\pm Q_n\}$ and $\{(E_n^\pm M_n^\sigma S P_n E_{-n}^\pm Q_n)^*\}$ are strongly convergent on ℓ^2.*

Proof. The uniform boundedness of these sequences is obvious. So it remains to prove their convergence on the elements $e_m = \{\delta_{mk}\}_{k=0}^\infty$ of the standard basis of ℓ^2. For $n > m \geq 1$, one has

$$f_{mn} := E_n^+ M_n^\sigma S P_n E_{-n}^+ Q_n e_{m-1} = \left\{ S \widetilde{\ell}_{mn}^\sigma (x_{jn}^\sigma) \right\}_{j=1}^n$$

(here we identify $\{\xi_0,\dots,\xi_{n-1},0,\dots\}\in\ell^2$ with $\{\xi_0,\dots,\xi_{n-1}\}$). We compute, for $x\neq x^\sigma_{kn}$,

$$\begin{aligned} S\widetilde{\ell}^\sigma_{kn}(x) &= \sqrt{\frac{\pi}{2}}\frac{(-1)^{k+1}}{n}\frac{1}{\pi i}\int_{-1}^1 \frac{\varphi(y)T_n(y)}{(y-x^\sigma_{kn})(y-x)}\,dy = \\ &= \sqrt{\frac{\pi}{2}}\frac{(-1)^{k+1}}{n(x^\sigma_{kn}-x)}\frac{1}{\pi i}\int_{-1}^1\left(\frac{1}{y-x^\sigma_{kn}}-\frac{1}{y-x}\right)\varphi(y)T_n(y)\,dy \end{aligned}$$

and, taking into account (3.16),

$$\begin{aligned} \frac{1}{\pi}\int_{-1}^1\frac{1}{y-x}\,\varphi(y)T_n(y)\,dy &= \frac{1}{\pi}\int_{-1}^1\frac{1-y^2}{y-x}\,T_n(y)\sigma(y)\,dy = \\ &= (1-x^2)U_{n-1}(y)-\frac{1}{\pi}\int_{-1}^1(y+x)T_n(y)\sigma(y)\,dy = \\ &= (1-x^2)U_{n-1}(x)\,. \end{aligned}$$

Thus, for $j\neq k$,

$$(S\widetilde{\ell}^\sigma_{kn})(x^\sigma_{jn}) = \frac{1}{ni}\,\frac{\varphi(x^\sigma_{kn})-(-1)^{j+k}\varphi(x^\sigma_{jn})}{x^\sigma_{kn}-x^\sigma_{jn}} =: s^{(n)}_{jk}\,.$$

With the help of

$$\frac{d}{dx}\left[(1-x^2)U_{n-1}(x)\right] = (1-x^2)U'_{n-1}(x)-2xU_{n-1}(x)$$

we get

$$(S\widetilde{\ell}^\sigma_{kn})(x^\sigma_{kn}) = -\frac{1}{ni}\,\frac{x^\sigma_{kn}}{\varphi(x^\sigma_{kn})} =: s^{(n)}_{kk}\,.$$

It follows

$$s^{(n)}_{jk} = \begin{cases} -\dfrac{\cos\frac{k+j-1}{2n}\pi}{in\sin\frac{k+j-1}{2n}\pi}\,, & j+k \text{ even,}\\[2ex] -\dfrac{\cos\frac{k-j}{2n}\pi}{in\sin\frac{k-j}{2n}\pi}\,, & j+k \text{ odd,}\end{cases} \tag{6.3}$$

and, consequently,

$$|s^{(n)}_{jk}| \le \begin{cases} \dfrac{1}{k+j-1}\,, & j+k \text{ even,}\\[2ex] \dfrac{1}{|k-j|}\,, & j+k \text{ odd.}\end{cases}$$

Thus, for fixed m, the sequences $\{f_{mn}\}=\{s^{(n)}_{1m},\dots,s^{(n)}_{nm},0,\dots\}$ are uniformly dominated by a square summable sequence, which implies

$$f_{mn}\longrightarrow\left\{\lim_{n\to\infty}s^{(n)}_{jm}\right\}_{j=1}^\infty =: \{s_{jm}\}\quad\text{in}\quad \ell^2\,,$$

where $s_{jk}=\lim\limits_{n\to\infty}s^{(n)}_{jk}$, i. e.

$$s_{jk} = \left\{\begin{array}{ll} -\dfrac{2}{\pi i(j+k-1)}\,, & j+k \text{ even}\\[2ex] \dfrac{2}{\pi i(j-k)}\,, & j+k \text{ odd}\end{array}\right\} = \frac{1-(-1)^{j-k}}{\pi i(j-k)}-\frac{1-(-1)^{j+k-1}}{\pi i(j+k-1)}\,. \tag{6.4}$$

Thus,

$$E_n^+ M_n^\sigma S P_n E_{-n}^+ Q_n \longrightarrow \mathbf{S} := [s_{jk}]_{j,k=1}^\infty \quad \text{on} \quad \ell^2 .$$

Now it is easy to see that

$$(E_n^+ M_n^\sigma S P_n E_{-n}^+ Q_n)^* \longrightarrow \mathbf{S}^* = [\overline{s_{kj}}]_{j,k=1}^\infty \quad \text{on} \quad \ell^2 ,$$

where $\overline{s_{kj}} = \dfrac{1-(-1)^{j-k}}{\pi i(j-k)} + \dfrac{1-(-1)^{j+k-1}}{\pi i(j+k-1)}$. Hence, denoting by $T(a) = [a_{j-k}]_{j,k=0}^\infty$ and $H(a) = [a_{j+k+1}]_{j,k=0}^\infty$ the Toeplitz and Hankel operator w.r.t. the symbol $a(t) = \displaystyle\sum_{k=-\infty}^{\infty} a_k t^k$, $t \in \mathbb{T}$, respectively, we have

$$\mathbf{S} = T(\phi) - H(\phi) \quad \text{and} \quad \mathbf{S}^* = T(\phi) + H(\phi) \quad \text{with} \quad \phi(t) = \operatorname{sgn}(\Im t) .$$

Finally, from

$$s_{n+1-j,n+1-k}^{(n)} = \begin{cases} \dfrac{\cos\frac{k+j-1}{2n}\pi}{in\sin\frac{k+j-1}{2n}\pi}, & j+k \quad \text{even}, \\[2ex] -\dfrac{\cos\frac{j-k}{2n}\pi}{in\sin\frac{j-k}{2n}\pi}, & j+k \quad \text{odd}, \end{cases}$$

we get

$$E_n^- M_n^\sigma S P_n E_{-n}^- Q_n \longrightarrow -\mathbf{S} \quad \text{and} \quad (E_n^- M_n^\sigma S P_n E_{-n}^- Q_n)^* \longrightarrow -\mathbf{S}^*$$

in ℓ^2.

We remark that the assertion of the previous lemma is not directly used in the following, but it essentially suggests the further considerations.

For $k, n \in \mathbb{Z}$ and $n \ge 1$, let $\widetilde{\varphi}_k^n$ denote the characteristic function of the interval $\left[\dfrac{k}{n}, \dfrac{k+1}{n}\right)$ multiplied by $\sqrt{n}$. Then the operators

$$\widetilde{E}_n : \ell^2_{\mathbb{Z}} \longrightarrow \mathbf{L}^2(\mathbb{R}) , \quad \{\xi_k\}_{k=-\infty}^\infty \mapsto \sum_{k=-\infty}^{\infty} \xi_k \widetilde{\varphi}_k^n$$

and

$$\widetilde{E}_{-n} = (\widetilde{E}_n)^{-1} : \operatorname{im} \widetilde{E}_n \longrightarrow \ell^2_{\mathbb{Z}} , \qquad \sum_{k=-\infty}^{\infty} \xi_k \widetilde{\varphi}_k^n \mapsto \{\xi_k\}_{k=-\infty}^\infty .$$

act as isometries. If we further denote the orthogonal projection from $\mathbf{L}^2(\mathbb{R})$ onto $\operatorname{im} \widetilde{E}_n$ by $\widetilde{L}_n$, then we get as a consequence of [7], Prop. 2.10 and Exerc. E2.11, the following lemma.

Lemma 6.3. *The sequence* $\widetilde{E}_n \mathbf{S} \widetilde{E}_{-n} \widetilde{L}_n : \mathbf{L}^2(\mathbb{R}) \longrightarrow \mathbf{L}^2(\mathbb{R})$ *is strongly convergent.*

Lemma 6.4. *The sequences* $\{E_{-n}^\pm Q_n \mathbf{S} Q_n E_n^\pm P_n\}$ *belong to the algebra* $\mathcal{F}^W$.

Proof. Obviously, the sequences under consideration are uniformly bounded. For $k = 1, \ldots, n$, define functions

$$\varphi_k^n(x) = \begin{cases} \sqrt{\dfrac{n}{\pi}}, & \cos\dfrac{k\pi}{n} \le x < \cos\dfrac{k-1}{n}\pi, \\[2ex] 0, & \text{otherwise}, \end{cases}$$

and let $R_n, S_n : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$ refer to the operators

$$R_n f = \sqrt{\frac{n}{\pi}} \sum_{k=1}^n \langle f, \varphi_k^n \rangle_\sigma \widetilde{\ell}^\sigma_{kn} \,, \quad S_n f = \sum_{k=1}^n \langle f, \varphi_k^n \rangle_\sigma \varphi_k^n \,.$$

Then, in view of (6.2),

$$\|R_n f\|_\sigma^2 \le c_2 \sum_{k=1}^n |\langle f, \varphi_k^n \rangle_\sigma|^2 = c_2 \|S_n f\|_\sigma^2 \le c_2 \|f\|_\sigma^2 \,,$$

i. e. the sequence $\{R_n\} \subset \mathcal{L}(\mathbf{L}^2_\sigma)$ is uniformly bounded. Moreover, for the characteristic function $f = \chi_{[x,y]}$ of an interval $[x, y] \subset [-1, 1]$, we have

$$\left| \langle f, \varphi_k^n \rangle_\sigma - \sqrt{\frac{\pi}{n}} f(x_{kn}^\sigma) \right| = \left| \sqrt{\frac{n}{\pi}} \int_{\frac{k-1}{n}\pi}^{\frac{k\pi}{n}} \left[f(\cos s) - f\left(\cos \frac{2k-1}{2n}\pi \right) \right] ds \right| = 0,$$

$$x, y \notin \left(\cos \frac{k\pi}{n}, \cos \frac{k-1}{n}\pi \right) ;$$

$$\left| \langle f, \varphi_k^n \rangle_\sigma - \sqrt{\frac{\pi}{n}} f(x_{kn}^\sigma) \right| = \left| \sqrt{\frac{n}{\pi}} \int_{\frac{k-1}{n}\pi}^{\frac{k\pi}{n}} \left[f(\cos s) - f\left(\cos \frac{2k-1}{2n}\pi \right) \right] ds \right| \le \sqrt{\frac{\pi}{n}}, \quad \text{otherwise},$$

which implies, again by (6.2),

$$\|R_n f - M_n^\sigma f\|_\sigma^2 = \left\| \sqrt{\frac{n}{\pi}} \sum_{k=1}^n \left[\langle f, \varphi_k^n \rangle_\sigma - \sqrt{\frac{\pi}{n}} f(x_{kn}^\sigma) \right] \widetilde{\ell}^\sigma_{kn} \right\|_\sigma^2 \le c_2 \, \frac{2\pi}{n} \,.$$

Consequently, $R_n f \longrightarrow f$ in $\mathbf{L}^2_\sigma$ for all $f \in \mathbf{L}^2_\sigma$. In particular, we get the following equivalences ($\xi_k^n \in \mathbb{C}$):

$$\begin{aligned} \sum_{k=1}^n \xi_k^n \widetilde{\ell}^\sigma_{kn} \longrightarrow f \quad \text{in} \quad \mathbf{L}^2_\sigma \quad &\Leftrightarrow \quad \lim_{n\to\infty} \left\| \sum_{k=1}^n \xi_k^n \widetilde{\ell}^\sigma_{kn} - R_n f \right\|_\sigma = 0, \\ &\Leftrightarrow \quad \lim_{n\to\infty} \sum_{k=1}^n \left| \sqrt{\frac{\pi}{n}} \xi_k^n - \langle f, \varphi_k^n \rangle_\sigma \right|^2 = 0, \\ &\Leftrightarrow \quad \lim_{n\to\infty} \left\| \sum_{k=1}^n \sqrt{\frac{\pi}{n}} \xi_k^n \varphi_k^n - S_n f \right\|_\sigma = 0, \\ &\Leftrightarrow \quad \sum_{k=1}^n \sqrt{\frac{\pi}{n}} \xi_k^n \varphi_k^n \longrightarrow f \quad \text{in} \quad \mathbf{L}^2_\sigma \,. \end{aligned}$$

Since $R_n \longrightarrow I$ in $\mathbf{L}^2_\sigma$, the convergence $E^+_{-n} Q_n \mathbf{S} Q_n E^+_n P_n f \longrightarrow g$ in $\mathbf{L}^2_\sigma$ for an $f \in \mathbf{L}^2_\sigma$ is equivalent to

$$E^+_{-n} Q_n \mathbf{S} Q_n E^+_n R_n f = E^+_{-n} Q_n \mathbf{S} Q_n \{\langle f, \varphi_k^n \rangle_\sigma\}_{k=1}^n \longrightarrow g \quad \text{in} \quad \mathbf{L}^2_\sigma$$

and, due to the previous considerations, equivalent to

$$\sum_{j=1}^{n}\sum_{k=1}^{n} s_{jk}\langle f,\varphi_k^n\rangle_\sigma \varphi_j^n \longrightarrow g \quad \text{in} \quad \mathbf{L}^2_\sigma\,. \tag{6.5}$$

The mapping $T:\mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2(0,1)$ defined by $(Tf)(s)=\sqrt{\pi}\,f(\cos\pi s)$ is an isometry, whereby $T\varphi_k^n=\widetilde{\varphi}_k^n$. Consequently, (6.5) is equivalent to

$$\chi_{[0,1]}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} s_{jk}\langle Tf,\widetilde{\varphi}_k^n\rangle_{\mathbf{L}^2(\mathbb{R})}\widetilde{\varphi}_j^n \longrightarrow \chi_{[0,1]}Tg \quad \text{in} \quad \mathbf{L}^2(\mathbb{R})\,. \tag{6.6}$$

The left-hand side of (6.6) can be written as $\widetilde{E}_n\mathbf{S}\widetilde{E}_{-n}\widetilde{L}_n\chi_{[0,1]}Tf$, and Lemma 6.3 guarantees the convergence of this sequence. Hence, we have proved that $W\{A_n\}$ exists for $A_n=E_{-n}^+Q_n\mathbf{S}Q_nE_n^+P_n$.

To prove the existence of $\widetilde{W}\{A_n\}$, we proceed as follows. By definitions and by taking into account (3.11) and (3.18) we find, for $u\in\mathbf{L}^2_\sigma$ and $\xi\in\ell^2$,

$$\begin{aligned} E_n^+W_nu &= \left\{\sqrt{\frac{\pi}{n}}\sum_{j=0}^{n-1}\langle u,\widetilde{u}_j\rangle_\sigma \widetilde{u}_{n-1-j}(x_{kn}^\sigma)\right\}_{k=1}^{n} = \\ &= \left\{\sqrt{\frac{\pi}{n}}\sum_{j=0}^{n-1}\langle u,\widetilde{u}_j\rangle_\sigma (-1)^{k+1}\gamma_jT_j(x_{kn}^\sigma)\right\}_{k=1}^{n}, \end{aligned}$$

$$\widetilde{\ell}_{kn}^\sigma = M_n^\sigma\widetilde{\ell}_{kn}^\sigma = \frac{\pi}{n}\sum_{j=0}^{n-1}\varepsilon_{jn}\widetilde{u}_j(x_{kn}^\sigma)\widetilde{u}_j\,, \tag{6.7}$$

and

$$\begin{aligned} W_nE_{-n}^+Q_n\xi &= \sqrt{\frac{\pi}{n}}\sum_{k=1}^{n}\xi_{k-1}\sum_{j=0}^{n-1}\varepsilon_{n-1-j,n}\widetilde{u}_{n-1-j}(x_{kn}^\sigma)\widetilde{u}_j = \\ &= \sqrt{\frac{\pi}{n}}\sum_{k=1}^{n}\xi_{k-1}\sum_{j=0}^{n-1}\gamma_j(-1)^{k+1}T_j(x_{kn}^\sigma)\widetilde{u}_j = \\ &= J_\sigma^{-1}\sqrt{\frac{\pi}{n}}\sum_{k=1}^{n}\xi_{k-1}(-1)^{k+1}\sum_{j=0}^{n-1}T_j(x_{kn}^\sigma)T_j = \\ &= J_\sigma^{-1}\sqrt{\frac{n}{\pi}}\sum_{k=1}^{n}\xi_{k-1}(-1)^{k+1}\ell_{kn}^\sigma\,. \end{aligned}$$

Thus, if we define $P_n^\sigma:\mathbf{L}^2_\sigma\longrightarrow\mathbf{L}^2_\sigma$, $E_n^\sigma:\operatorname{im}P_n^\sigma\longrightarrow\operatorname{im}Q_n$, and $\widetilde{W}:\ell^2\longrightarrow\ell^2$ by

$$P_n^\sigma f=\sum_{k=0}^{n-1}\langle u,T_k\rangle_\sigma T_k\,,\quad E_n^\sigma f=\{f(x_{kn}^\sigma)\}_{k=1}^n\,,\quad \widetilde{W}\xi=\left\{(-1)^k\xi_k\right\}_{k=0}^\infty\,,$$

respectively, then $E_nW_n=\widetilde{W}E_n^\sigma J_\sigma P_n=\widetilde{W}E_n^\sigma P_n^\sigma J_\sigma$ and $W_nE_{-n}^+=J_\sigma^{-1}E_{-n}^\sigma\widetilde{W}$, where $E_{-n}^\sigma:=(E_n^\sigma)^{-1}$. Consequently,

$$W_nE_{-n}^+Q_n\mathbf{S}Q_nE_n^+W_n=J_\sigma^{-1}E_{-n}^\sigma Q_n\widetilde{W}\mathbf{S}\widetilde{W}Q_nE_n^\sigma P_n^\sigma J_\sigma. \tag{6.8}$$

The strong convergence of this sequence can be proved as the strong convergence of the operators $E_{-n}^+ Q_n \mathbf{S} Q_n E_n^+ P_n$.

Defining $J_n u = \sum_{j=0}^\infty \varepsilon_{jn} \langle u, \widetilde{u}_j \rangle_\sigma \widetilde{u}_j$ ($\varepsilon_{jn} := 1$, $j > n-1$) and taking into account (6.7), we can write, for $u \in \mathbf{L}_\sigma^2$ and $\xi \in \ell^2$,

$$\begin{aligned} \langle Q_n E_n^+ P_n J_n u, \xi \rangle_{\ell^2} &= \sqrt{\frac{\pi}{n}} \sum_{k=1}^n \sum_{j=0}^{n-1} \varepsilon_{jn} \langle u, \widetilde{u}_j \rangle_\sigma \widetilde{u}(x_{kn}^\sigma) \overline{\xi}_{k-1} = \\ &= \langle u, \sqrt{\frac{n}{\pi}} \sum_{k=1}^n \xi_{k-1} \frac{\pi}{n} \sum_{j=0}^{n-1} \varepsilon_{jn} \widetilde{u}_j(x_{kn}^\sigma) \widetilde{u}_j \rangle_\sigma = \\ &= \langle u, E_{-n}^+ Q_n \xi \rangle_\sigma , \end{aligned}$$

which leads to $(Q_n E_n^+ P_n J_n)^* = E_{-n}^+ Q_n$, i.e. $(Q_n E_n^+ P_n)^* = J_n^{-1} E_{-n}^+ Q_n$. Furthermore, again due to (6.7),

$$\begin{aligned} \langle E_{-n}^+ Q_n \xi, u \rangle_\sigma &= \sqrt{\frac{n}{\pi}} \sum_{k=1}^n \xi_{k-1} \langle \widetilde{\ell}_{kn}^\sigma, P_n u \rangle_\sigma = \\ &= \sqrt{\frac{\pi}{n}} \sum_{k=1}^n \xi_{k-1} \sum_{j=0}^{n-1} \overline{\langle u, \widetilde{u}_j \rangle_\sigma} \varepsilon_{jn} \widetilde{u}_j(x_{kn}^\sigma) = \\ &= \langle \xi, E_n^+ P_n J_n u \rangle_{\ell^2} , \end{aligned}$$

such that $(E_{-n}^+ Q_n)^* = E_n^+ P_n J_n$. Hence,

$$(E_{-n}^+ Q_n \mathbf{S} Q_n E_n^+ P_n)^* = J_n^{-1} E_{-n}^+ Q_n \mathbf{S}^* Q_n E_n^+ P_n J_n .$$

Analogously, with the help of (6.8) one can show that

$$(W_n E_{-n}^+ Q_n \mathbf{S} Q_n E_n^+ W_n)^* = J_\sigma^* E_{-n}^\sigma Q_n \widetilde{W} \mathbf{S}^* \widetilde{W} Q_n E_n^\sigma P_n^\sigma J_n^{-*} .$$

Since $J_n \longrightarrow I$ in $\mathbf{L}_\sigma^2$ we conclude also the strong convergence of $A_n^* P_n$ and $(W_n A_n W_n)^* P_n$, and the proof of $\left\{ E_{-n}^+ Q_n \mathbf{S} Q_n E_n^+ P_n \right\} \in \mathcal{F}^W$ is done.

For the second sequence we can use the same arguments taking into account the following two facts:

a) $E_n^- = V_n E_n^+$ and $E_{-n}^- = E_{-n}^+ V_n$, where $V_n \xi = \{\xi_{n-1}, \ldots, \xi_0, 0, \ldots\}$, $\xi \in \ell^2$. Consequently, $E_{-n}^- Q_n \mathbf{S} Q_n E_n^- P_n = E_{-n}^+ V_n \mathbf{S} V_n E_n^+ P_n$;

b) $V_n T(a) V_n = Q_n T(\widetilde{a}) Q_n$, where $\widetilde{a} = a(t^{-1})$, and $H(\phi)$ belongs to the smallest closed subalgebra $\mathcal{T}$ of $\mathcal{L}(\ell^2)$ containing all Toeplitz operators $T(a)$ with $a \in \mathbf{PC}(\mathbb{T})$. Thus, $V_n \mathbf{S} V_n = Q_n \mathbf{S}_0 Q_n$, $\mathbf{S}_0 \in \mathcal{T}$, and Lemma 6.3 remains true for $\mathbf{S}_0$ instead of $\mathbf{S}$.

This completes the proof of the lemma.

In what follows we will use the local principle of Gohberg and Krupnik (see below). Although it is possible to apply the local principle of Allan and Douglas (cf. Section 4) equivalently, we decided to go this other way with the aim of a little more clear presentation.

Let $\mathcal{B}$ be a unital Banach algebra. A subset $\mathcal{M} \subset \mathcal{B}$ is called a localizing class if $0 \notin \mathcal{M}$ and if, for all $a_1, a_2 \in \mathcal{M}$, there exists an element $a \in \mathcal{M}$ such that

$$aa_j = a_j a = a \quad \text{for} \quad j = 1, 2 .$$

Let $\mathcal{M}$ be a localizing class. Two elements $x,\, y \in \mathcal{B}$ are called $\mathcal{M}$-equivalent (in symbols: $x \overset{\mathcal{M}}{\sim} y$), if

$$\inf_{a \in \mathcal{M}} \|a(x-y)\|_{\mathcal{B}} = \inf_{a \in \mathcal{M}} \|(x-y)a\|_{\mathcal{B}} = 0 .$$

Further, $x \in \mathcal{B}$ is called $\mathcal{M}$-invertible if there exist $a_1, a_2 \in \mathcal{M}$ and $z_1, z_2 \in \mathcal{B}$ such that

$$z_1 x a_1 = a_1 \quad \text{and} \quad a_2 x z_2 = a_2 .$$

A system $\{\mathcal{M}_\tau\}_{\tau \in \Omega}$ of localizing classes (Ω is an arbitrary index set) is said to be covering if, for each system $\{a_\tau\}_{\tau \in \Omega}$ with $a_\tau \in \mathcal{M}_\tau$, there exists a finite subsystem $a_{\tau_1}, \ldots, a_{\tau_n}$ such that $a_{\tau_1} + \cdots + a_{\tau_n}$ is invertible in the algebra $\mathcal{B}$.

Local principle of Gohberg and Krupnik ([6], Theorem XII.1.1). Let $\mathcal{B}$ be a unital Banach algebra, $\{\mathcal{M}_\tau\}_{\tau \in \Omega}$ a covering system of localizing classes in $\mathcal{B}$, $x \in \mathcal{B}$ and $x \overset{\mathcal{M}_\tau}{\sim} x_\tau$ for all $\tau \in \Omega$. Then x is $\mathcal{M}_\tau$-invertible if and only if x_τ is $\mathcal{M}_\tau$-invertible. If x commutes with all elements from $\bigcup_{\tau \in \Omega} \mathcal{M}_\tau$, then x is invertible in $\mathcal{B}$ if and only if x_τ is $\mathcal{M}_\tau$-invertible for all $\tau \in \Omega$.

For $\tau \in [-1,1]$, let

$$m_\tau := \{f \in \mathbf{C}[-1,1] : 0 \le f(x) \le 1 ,\ f(x) \equiv 1 \text{ in some neighborhood of } \tau\}$$

and define $\mathcal{M}_\tau := \{\{M_n^\sigma f P_n\}^o : f \in m_\tau\}$. Then $\{\mathcal{M}_\tau\}_{\tau \in [-1,1]}$ forms a covering system of localizing classes in $\mathcal{F}^W/\mathcal{J}$, which, due to Lemma 4.1, has the property that all elements of this system commute with all elements of the form $\{M_n^\sigma(aI + bS)P_n\}^o$, $a, b \in \mathbf{PC}$. The Relation (3.5) shows that, for $a, a_1, b, b_1 \in \mathbf{PC}$, the cosets $\{M_n^\sigma(aI + bS)P_n\}^o$ and $\{M_n^\sigma(a_1 I + b_1 S)P_n\}^o$ are $\mathcal{M}_\tau$-equivalent if $a(\tau \pm 0) = a_1(\tau \pm 0)$ and $b(\tau \pm 0) = b_1(\tau \pm 0)$.

Lemma 6.5. *The cosets* $\{M_n^\sigma S P_n\}^o$ *and* $\left\{\pm E_{-n}^\pm Q_n \mathbf{S} Q_n E_n^\pm P_n\right\}^o$ *are* $\mathcal{M}_{\pm 1}$*-equivalent.*

Proof. Let $a \in m_1$ and $B_n = E_n^+ M_n^\sigma a P_n E_{-n}^+ (E_n^+ M_n^\sigma S P_n E_{-n}^+ Q_n - Q_n \mathbf{S} Q_n)$. Then

$$B_n = \operatorname{diag}[a(x_{1n}^\sigma), \ldots, a(x_{nn}^\sigma)][s_{jk}^{(n)} - s_{jk}]_{j,k=1}^n$$

and, due to the uniform boundedness of the sequences $\{E_n^+\}$ and $\left\{E_{-n}^+\right\}$,

$$\left\|M_n^\sigma a P_n (M_n^\sigma S P_n - E_{-n}^+ Q_n \mathbf{S} Q_n E_n^+ P_n)\right\|_{\mathcal{L}(\mathbf{L}_\sigma^2)} \le \text{const } \|B_n\|_{\mathcal{L}(\ell^2)} .$$

Assume that $\operatorname{supp} a \subset [\cos \pi\varepsilon, 1]$, $0 < \varepsilon < 1/4$. Since the function $g(z) = z \cot z = 1 - \dfrac{1}{3} z^2 + \cdots$ is analytic for $|z| < \pi$, we have $\left|\cot z - z^{-1}\right| \le \text{const}\,|z|$ for $|z| \le 3\pi/4$, which leads to $\left|s_{jk}^{(n)} - s_{jk}\right| \le \text{const}\, \dfrac{k+j}{n^2}$. It follows

$$\|B_n\|_{\mathcal{L}(\ell^2)}^2 \le \sum_{1 \le j \le n\varepsilon + \frac{1}{2}} \sum_{k=1}^n |s_{jk}^{(n)} - s_{jk}|^2 \le \text{const} \left(\varepsilon + \frac{1}{n}\right)$$

and, consequently,

$$\left\|\left\{M_n^\sigma a P_n(M_n^\sigma S P_n - E_{-n}^+ Q_n \mathbf{S} Q_n E_n^+ P_n)\right\}^o\right\| \le \mathrm{const}\,\sqrt{\varepsilon}\,.$$

Analogously one can show that

$$\left\|\left\{(M_n^\sigma S P_n - E_{-n}^+ Q_n \mathbf{S} Q_n E_n^+ P_n) M_n^\sigma a P_n\right\}^o\right\| \le \mathrm{const}\,\sqrt{\varepsilon}\,,$$

and the $\mathcal{M}_1$-equivalence of $\{M_n^\sigma S P_n\}^o$ and $\left\{E_{-n}^+ Q_n \mathbf{S} Q_n E_n^+\right\}^o$ is proved. The proof of the $\mathcal{M}_{-1}$-equivalence of $\{M_n^\sigma S P_n\}^o$ and $\left\{-E_{-n}^- Q_n \mathbf{S} Q_n E_n^-\right\}^o$ is similar.

Lemma 6.6. *The sequence* $\left\{E_{-n}^\pm Q_n \mathbf{S} Q_n E_n^\pm P_n - \lambda P_n\right\}$ *is stable in* $\mathbf{L}^2_\sigma$ *if and only if* $\lambda \notin \mathbb{D}_+ := \{z \in \mathbb{C} : |z| \le 1,\, \Im z \ge 0\}$.

Proof. Due to [2, Prop. 4.1], the sequence $\{Q_n \mathbf{S}^* Q_n - \lambda O_n\}$ is stable in ℓ^2 if and only if $\lambda \notin \mathbb{D}_- := \{z \in \mathbb{C} : |z| \le 1,\, \Im z \le 0\}$. This fact implies the assertion immediately (recall the uniform boundedness of $E_n^\pm$ and $E_{-n}^\pm = (E_n^\pm)^{-1}$).

Proof of Lemma 4.4. Let $\tau = \pm 1$. Lemma 3.5 and the local principle of Allan and Douglas imply that $\sigma_{(\mathcal{G}/\mathcal{K})/\mathcal{J}_\tau^{\mathcal{G}}}(S) = \mathbb{T}_\tau$. Further, by Lemmas 4.3 and 6.1,

$$\mathbb{T}_\tau \subset \sigma_{(\mathcal{A}^\sigma/\mathcal{J})/\mathcal{J}_\tau^\sigma}\left(\{M_n^\sigma S P_n\}^o + \mathcal{J}_\tau^\sigma\right) \subset \mathbb{T}\,.$$

Let $\lambda \in \mathbb{T}\setminus\mathbb{T}_\tau$. Due to Lemmas 6.6 and 6.4, the coset $\left\{\pm E_{-n}^\pm Q_n \mathbf{S} Q_n E_n^\pm P_n - \lambda P_n\right\}^o$ is invertible in $\mathcal{F}^W/\mathcal{J}$. By Lemma 6.5 and the local principle of Gohberg and Krupnik we get the $\mathcal{M}_\tau$-invertibility of $\{M_n^\sigma S P_n - \lambda P_n\}^o$.

Let $\chi(x) = \dfrac{1+x}{2}$ and $\lambda \in \mathbb{T}\setminus\mathbb{T}_1$. Then $\{M_n^\sigma \chi S P_n - \lambda P_n\}^o$ is $\mathcal{M}_1$-equivalent to $\{M_n^\sigma S P_n - \lambda P_n\}^o$, and $\mathcal{M}_{-1}$-equivalent to $\lambda\{P_n\}^o$. So, $\mathcal{M}_1$- and $\mathcal{M}_{-1}$-invertible. For $\tau \in (-1,1)$ we use the fact that $(\mathcal{A}^\sigma/\mathcal{J})/\mathcal{J}_\tau^\sigma$ is *-isomorphic to a C^*-algebra of continuous 2×2 matrix functions on $[0,1]$, which was shown in Section 4. This isomorphism sends

$$\{M_n^\sigma \chi S P_n - \lambda P_n\}^o + \mathcal{J}_\tau^\sigma = \left\{\frac{1+\tau}{2} M_n^\sigma S P_n - \lambda P_n\right\}^o + \mathcal{J}_\tau^\sigma$$

into the function

$$\mu \mapsto \begin{bmatrix} \dfrac{1+\tau}{2} - \lambda & 0 \\ 0 & -\dfrac{1+\tau}{2} - \lambda \end{bmatrix},$$

which is invertible. Consequently, for each $\tau \in (-1,1)$, there exist $\{B_n^\tau\} \in \mathcal{A}^\sigma$ and $\left\{T_{n,k}^\tau\right\}^o \in \mathcal{J}_\tau^\sigma$, $k = 1,2$, such that

$$\{B_n^\tau\}^o \{M_n^\sigma \chi S P_n - \lambda P_n\}^o = \{P_n\}^o + \left\{T_{n,1}^\tau\right\}^o$$

and

$$\{M_n^\sigma \chi S P_n - \lambda P_n\}^o \{B_n^\tau\}^o = \{P_n\}^o + \left\{T_{n,2}^\tau\right\}^o\,.$$

Since $\left\{T_{n,k}^\tau\right\}^o$ is $\mathcal{M}_\tau$-equivalent to the zero element of $\mathcal{A}^\sigma/\mathcal{J}$, we get the $\mathcal{M}_\tau$-invertibility of $\{M_n^\sigma \chi S P_n - \lambda P_n\}^o$ also for $\tau \in (-1,1)$. The local principle of Gohberg and Krupnik gives the invertibility of $\{M_n^\sigma \chi S P_n - \lambda P_n\}^o$ in $\mathcal{F}^W/\mathcal{J}$. Because of the inverse closedness of C^*-subalgebras, the inverse of $\{M_n^\sigma \chi S P_n - \lambda P_n\}^o$ belongs to $\mathcal{A}^\sigma/\mathcal{J}$, which implies, due to the local principle of Allan and Douglas, the invertibility of $\{M_n^\sigma \chi S P_n - \lambda P_n\}^o + \mathcal{J}_1^\sigma$.

The invertibility of $\{M_n^\sigma \chi S P_n - \lambda P_n\}^o + \mathcal{J}_{-1}^\sigma$ for $\lambda \in \mathbb{T}\setminus\mathbb{T}_{-1}$ can be shown analogously.

References

[1] Allan G. R. Ideals of vector valued functions // Proc. London Math. Soc. 1968. Vol. 18, No. 3. P. 193–216.

[2] Basor E. L., Ehrhardt T. On a class of Toeplitz + Hankel operators // New York J. Math. 1999. No. 5. P. 116.

[3] Coburn L. A. The C^*-algebra generated by an isometry // Bull. Amer. Math. Soc. 1967. No. 73. P. 722–726.

[4] Douglas R. G. Banach Algebra Techniques in Operator Theory. N. Y.: Acad. Press, 1972.

[5] Freud G. Orthogonale Polynome. Berlin: Deutscher Verlag der Wissenschaften, 1969.

[6] Gohberg I., Krupnik N. One-dimensional Linear Singular Integral Equations. Basel, Boston, Berlin: Birkhäuser Verlag, 1992.

[7] Hagen R., Roch S., Silbermann B. Spectral Theory of Approximation Methods for Convolution Equations. Basel, Boston, Berlin: Birkhäuser Verlag, 1994.

[8] Halmos P. R. Two subspaces // Trans. Amer. Math. Soc. 1969. No. 144. P. 381–389.

[9] Junghanns P., Müller K. A collocation method for nonlinear Cauchy singular integral equations // J. Comp. Appl. Math. 2000. No. 115. P. 283–300.

[10] Junghanns P., Silbermann B. Local theory of the collocation method for the approximate solution of singular integral equations // Integral Equations Operator Theory. 1984. No. 7. P. 791–807.

[11] Junghanns P., Weber U. Banach Algebra Techniques for Cauchy Singular Integral Equations on an Interval // Boundary Element Technology XII / Eds J. I. Frankel, C. A. Brebbia, M. A. H. Aliabadi. Boston: Comput. Mech. Publ., Southampton, 1997. P. 419–428.

[12] Junghanns P., Weber U. Local Theory of Projection Methods for Cauchy Singular Integral Equations on an Interval // Boundary Integral Methods: Numer. and Math. Aspects / Ed. M. A. Golberg. Boston: Comput. Mech. Publ. Ser.: Comput. Eng., Southampton, 1998. P. 217–256.

[13] Mikhlin S. G., Prössdorf S. Singular Integral Operators. Berlin: Akademie-Verlag, 1986.

[14] Nevai P. Orthogonal Polynomials // Mem. Amer. Math. Soc. 1979. Vol. 213.

[15] Pedersen G. K. C^*-Algebras and Their Automorphism Groups. London, N. Y., San Francisco: Academic Press, 1979.

[16] Prössdorf S., Silbermann B. Numerical Analysis for Integral and Related Operator Equations. Basel, Boston, Berlin: Birkhäuser Verlag, 1991.

[17] Roch S. Algebras of approximation sequences: Fractality. Preprint 2052(1999). TU Darmstadt (Proc. of the 11. TMP Conf. Chemnitz, 1999).

[18] Roch S. Algebras of approximation sequences: Fredholmness. Preprint 2048(1999). TU Darmstadt (to appear in the J. Oper. Theory).

[19] Roch S., Silbermann B. C^*-algebra techniques in numerical analysis // J. Oper. Theory. 1996. Vol. 35, No. 2. P. 241–280.

[20] Roch S., Silbermann B. Index calculus for approximation methods and singular value decomposition // J. Math. Anal. Appl. 1998. Vol. 225. P. 401–426.

[21] Silbermann B. Lokale Theorie des Reduktionsverfahrens für Toeplitzoperatoren // Math. Nachr. 1981. Vol. 104. P. 137–146.

[22] Weber U. Weighted polynomial approximation methods for Cauchy singular integral equations in the non-periodic case: PhD thesis. TU Chemnitz, 1999.

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