BEST PINSKER BOUND EQUALS TAYLOR POLYNOMIAL OF DEGREE 49*

A. A. Fedotov
Institute of Computational Technologies SB RAS, Novosibirsk, Russia
e-mail: `lesha@ict.nsc.ru`

P. Harremoës, F. Topsøe
University of Copenhagen, DK-2100 Copenhagen, Denmark
e-mail: `moes@math.ku.dk, topsoe@math.ku.dk`

Оценки Пинскера — это рекурсивно определяемые полиномы $P_\nu(V)$, для которых выполняется неравенство $L(V) \geq P_\nu(V)$, где $L(V)$ — это *точная граница Вайды.* Мы покажем, что $P_\nu(V)$ совпадает с полиномом Тейлора степени ν для $L(V)$ в нуле тогда и только тогда, когда ν не превосходит 49.

Introduction

Let us consider the set of probability distributions $M_+^1(n)$ over an alphabet with n letters. We introduce two measures of difference between two arbitrary elements P and Q from $M_+^1(n)$. We denote by $D = D\,(P \parallel Q) : M_+^1(n) \times M_+^1(n) \to [\,0;\, +\infty\,]$ the divergence from P to Q

$$D\,(P \parallel Q) = \sum_{i\in A} p_i \log \frac{p_i}{q_i},$$

and by $V = V(P, Q) : M_+^1(n) \times M_+^1(n) \to [\,0;\, 2\,]$ the total variation

$$V\,(P,\, Q) = \sum_{i\in A} |p_i - q_i|\,.$$

The former is a well-known instrument of Information Theory, which provides a nice geometrical approach of thinking about channel capacity and optimal predictors and the latter is a common functional analysis norm.

We are interested in lower bounds of D in terms of V. The research in this direction starts with an inequality by Pinsker [1], which was improved by Csiszár [2], Kemperman [3, 4], Kullback [5, 6] to

$$D \geq \frac{1}{2}V^2,$$

*The paper is supported by INTAS grant 00-738, by a post-doc grant from the Villum Kann Rasmussen Foundation and by the Danish Natural Science Research Council.

and the constant 1/2 was proved to be the best possible. The best two-term inequality of this type is

$$D \geq \frac{1}{2}V^2 + \frac{1}{36}V^4,$$

as proved by Krafft [7].

A further term $c_6\,V^6$ was added by Krafft and Schmitz [7], Toussaint [8] and by Topsøe [9], who proved that $c_6 = 1/270$ is the best possible constant. The authors have extended this result up to 8 terms in their recent work [10].

Here and further by the *best constants* $c_\nu^{\max}$, $\nu = 0,\ 1,\ 2,\ \ldots$, we shall understand the constants defined recursively by taking $c_\nu^{\max}$ to be the largest constant c for which the inequality

$$D \geq \sum_{i<\nu} c_i^{\max} V^i + cV^\nu \tag{1}$$

holds generally. Clearly, $c_\nu^{\max}$ are well defined non-negative real constants. In this work we call the polynomial from the right part of the inequality

$$P_\nu(V) = \sum_{i\leq\nu} c_i^{\max} V^i$$

as *refined Pinsker bound* and our main goal is to find out when it could be easily calculated.

Vajda [11] suggested a closer study of the function L defined by

$$L\left(V_0\right) = \inf_{V(P,Q)=V_0} D\left(P \parallel Q\right) \;\text{ for } V_0 \in [\,0;\,2\,[\,.$$

This function we shall refer to as *Vajda's tight lower bound*. In the work [10] we have obtained a parametrization of $L(V)$ expressed in terms of elementary functions, mainly the hyperbolic functions. We formulate this basic fact in the following theorem.

Theorem 1. *Diffeomorphisms* $V(t) : \mathbb{R}_+ \to [\,0;\,2\,[$, $L(t) : \mathbb{R}_+ \to \mathbb{R}_+$ *give the following parametrization of the tight lower bound* $L(V)$*:*

$$\begin{aligned} V\left(t\right): \quad t &\;\mapsto\; t\left(1-\left(\coth\left(t\right)-\frac{1}{t}\right)^2\right), \\ L(t): \quad t &\;\mapsto\; \log\left(\frac{t}{\sinh\left(t\right)}\right)+t\coth\left(t\right)-\frac{t^2}{\sinh^2\left(t\right)}. \end{aligned} \tag{2}$$

In the theorem and further we denote by $\mathbb{R}_+$ a set of non-negative real numbers and assume that our formulae continuously extend to zero singularities. The following corollary is a consequence of the fact that the parameter t is equal to dL/dV.

Corollary 1 (integral representation). *For all values* $V_0 \in [\,0;\,2\,[$ *Vajda's tight lower bound* $L(V)$ *can be written as*

$$L\left(V_0\right) = \int_0^{V_0} t\left(V\right)\,dV, \tag{3}$$

where $t\left(V\right) : [\,0;\,2\,[\to \mathbb{R}_+$ *is the inverse of the diffeomorphism* $V(t)$*.*

Exploring this parametrization one could note that any refined Pinsker bound $P_\nu(V)$ obtained by predesessors equals to Taylor polynomial $T_\nu(V)$ of degree ν constructed from $L(V)$ in a zero neighborhood. It would be nice to understand when they coincide because Taylor polynomials have a bunch of nice properties: their odd coefficients vanish because parametrization of $L(V)$ extends to an even function, and even coefficients of these polynomials are rational and could be easily calculated.

In our preceding paper [10] we found out that Taylor coefficient to V^{62} of $L(V)$ is negative, and therefore Taylor polynomials do not generally give a lower bound for $L(V)$. In this paper we discuss in detail whether the lower degree Taylor polynomial is a lower bound for $L(V)$. Shortly speaking, the first 50 polynomials provide the lower bound for L, and polynomials of degree from 50 till 95 do not.

Our main result could be readily deduced from this analysis. It states that refined Pinsker bounds correspond to Taylor polynomials of $L(V)$ iff their degree is 49 or less.

The proofs of these results are quite computational, and we were forced to use mathematical software. Under assumption that Waterloo Maple has a proper implementation of symbolic and interval arithmetic our proofs are strict. We collect proof ideas in the following section postponing all technical computations to the appendix.

1. Main Result

Let us start with some definitions. We denote by t_ν the coefficient of V^ν in Taylor expansion of $L(V)$ around zero and by $T_\nu(V)$ the Taylor polynomial of degree ν,

$$T_\nu(V) = \sum_{i=0}^{\nu} t_i\, V^i.$$

Let $R_\nu(V)$ be the difference between $L(V)$ and $T_\nu(V)$ that is $L(V) = T_\nu(V) + R_\nu(V)$ holds. In interval computations we use notation $[a..b]$ for some number from the range $[\,a;\, b\,]$.

The following theorem describes when Taylor polynomials of $L(V)$ are global lower bounds.

Theorem 2. *Consider the Taylor polynomial $T_\nu(V)$ of degree ν approximating $L(V)$ in zero. The inequality $L(V) \geq T_\nu(V)$ holds for all V from $[\,0;\, 2\,[$ if $\nu = 0,\, 1,\, \ldots,\, 49$. This inequality fails for some V if $\nu = 50,\, 51,\, \ldots,\, 95$.*

Proof. Computations in interval arithmetic give us

$$R_{50}(V([9.9212..9.9213])) = [-0.0041.. - 0.0034] < 0.$$

Hence $L(V) > T_{50}(V)$ does not hold for some V in a vicinity of $t = 10$. Due to Lemma 1 from Appendix we know that Taylor coefficients for $\nu = 52,\, 53,\, \ldots,\, 61$ are nonnegative and therefore the inequality $L(V) > T_\nu(V)$ does not hold for these ν either.

The lemma also states that for $\nu = 60,\, 61,\, \ldots,\, 95$ polynomials $T_\nu(V)$ are followed by the negative term in the Taylor expansion of $L(V)$. This negative term becomes the leading term of the difference $R_\nu(V)$, and therefore $R_\nu(V)$ is negative in a neighborhood of zero.

The difficult fact $L(V) \geq T_{49}$ is postponed to Lemma 3 in Appendix. To complete the proof we show that for $\nu \leq 49$ the following chain holds:

$$L(V) \geq T_{49}(V) \geq T_\nu(V).$$

It is obvious consequence of Lemma 1 which claims that coefficients t_ν for $\nu \leq 49$ are nonnegative. ■

From the theorem we can readily deduce our main result which makes a correspondence between refined Pinsker bounds and Taylor polynomials.

Theorem 3. *Refined Pinsker bound $P_\nu(V)$ is equal to the Taylor polynomial $T_\nu(V)$ of $L(V)$ in zero iff $n = 0, 1, \ldots, 49$.*

Proof. First, let us argue why $T_\nu(V)$ cannot be refined Pinsker bound for $\nu \geq 50$. Theorem 2 gives us that for ν less than 62 polynomial T_ν is not a lower bound. For ν greater or equal to 62 this polynomial contains a negative coefficient t_{62} according to Lemma 1, which cannot appear in a refined bound.

Then we need to prove that $T_\nu(V)$ is a refined Pinsker bound for $\nu < 50$. From Theorem 2 we obtain that $T_\nu(V)$ is a lower bound. Hence we only have to show that t_ν is a *best possible* constant, i. e. it cannot be substituted with a greater constant preserving the inequality

$$L(V) \geq \tilde{T}_\nu(V) = T_\nu(V) + \epsilon\, V^\nu.$$

Let us assume the substitution takes place. In this case the leading coefficient of the difference become negative:

$$\tilde{R}_\nu(V) = L(V) - \tilde{T}_\nu(V) = -\epsilon\, V^\nu + R_\nu(V).$$

Thus $R_\nu(V)$ is a Taylor reminder term, $R_\nu(V) = o\left(V^\nu\right)$ in a neighborhood of zero and therefore $\tilde{R}_\nu(V)$ takes negative values in a neighborhood of zero. Hence the inequality fails, and our assumption fails also. This finishes the proof. ■

2. Appendix

We have placed here all the computational things which could not be avoided if we want to present strict proofs. Nevertheless among the bulky formulae one could find some usefull techniques how to deal with Waterloo Maple or any other software suited for symbolic computations.

The following lemma collects the properties we need to know about the Taylor series of $L(V)$:

Lemma 1. *Consider the first 98 terms of the Taylor series of $L(V)$:*

$$L(V) = \sum_{\nu=0}^{97} t_\nu\, V^\nu + O\left(V^{98}\right), \quad V \to 0.$$

We claim the following facts about Taylor coefficients t_ν.

1. *All odd coefficients vanish.*

2. *Coefficients $t_2, t_4, \ldots, t_{60}$ are positive.*

3. *Coefficients $t_{62}, t_{64}, \ldots, t_{96}$ are negative.*

Proof. We establish these facts by a straightforward computation of the decomposition.

$$
\begin{aligned}
L(V) \;=\; & \frac{1}{2}V^2 + \frac{1}{2^2\,3^2}V^4 + \frac{1}{2\,3^3\,5}V^6 + \frac{221}{2^3\,3^5\,5^2\,7}V^8 + \frac{299}{2\,3^8\,5^2\,7}V^{10} \\
+\; & \frac{5983}{2^2\,3^9\,5\,7^2\,11}V^{12} + \frac{9953639}{2\,3^{11}\,5^4\,7^2\,11\,13}V^{14} + \frac{24080603}{2^4\,3^{13}\,5^4\,7\,11\,13}V^{16} \\
+\; & \frac{258692351}{2\,3^{14}\,5^4\,7^2\,11\,13\,17}V^{18} + \frac{125041974165263}{2^2\,3^{17}\,5^6\,7^3\,11^2\,13\,17\,19}V^{20} \\
+\; & \frac{195059968637159}{2\,3^{19}\,5^5\,7^4\,11^2\,13\,17\,19}V^{22} + \frac{79414742287586653}{2^3\,3^{20}\,5^6\,7^4\,11\,13^2\,17\,19\,23}V^{24} \\
+\; & \frac{12332430212594640377}{2\,3^{22}\,5^8\,7^4\,11^2\,13^2\,17\,19\,23}V^{26} + \frac{386905591728850339030}{2^2\,3^{26}\,5^8\,7^4\,11^2\,13\,17\,19\,23}V^{28} \\
+\; & \frac{1102997556766204706333}{2\,3^{27}\,5^7\,7^4\,11^2\,13^2\,17\,19\,23\,29}V^{30} \\
+\; & \frac{142080867274907112114975344 6087}{2^5\,3^{29}\,5^{10}\,7^6\,11^3\,13^2\,17^2\,19\,23\,29\,31}V^{32} \\
+\; & \frac{2009255808369959828225697 36751}{2\,3^{31}\,5^{10}\,7^6\,11^3\,13^2\,17^2\,19\,23\,29\,31}V^{34} \\
+\; & \frac{16874810741641816631426723 43299}{2^2\,3^{32}\,5^{11}\,7^6\,11^3\,13^2\,17\,19^2\,23\,29\,31}V^{36} \\
+\; & \frac{2552634697578168238912697690522568337}{2\,3^{35}\,5^{12}\,7^6\,11^4\,13^3\,17^2\,19^2\,23\,29\,31\,37}V^{38} \\
+\; & \frac{8374929973147573235437437941603041543}{2^3\,3^{37}\,5^{12}\,7^7\,11^4\,13^3\,17^2\,19\,23\,29\,31\,37}V^{40} \\
+\; & \frac{16788450705527362554234490440283671678 13}{2\,3^{38}\,5^{11}\,7^8\,11^4\,13^3\,17^2\,19^2\,23\,29\,31\,37\,41}V^{42} \\
+\; & \frac{1292424096115078298883603771212276425159 15757}{2^2\,3^{40}\,5^{14}\,7^7\,11^4\,13^3\,17^2\,19^2\,23^2\,29\,31\,37\,41\,43}V^{44} \\
+\; & \frac{6884123208987742399990419281013274456650 72803}{2\,3^{43}\,5^{13}\,7^8\,11^3\,13^4\,17^2\,19^2\,23^2\,29\,31\,37\,41\,43}V^{46} \\
+\; & \frac{867977208074847239122838099254943590347978 61067}{2^4\,3^{44}\,5^{13}\,7^8\,11^4\,13^4\,17^2\,19^2\,23\,29\,31\,37\,41\,43\,47}V^{48} \\
+\; & \frac{16717278215656887686015556945540739970787015266791939}{2\,3^{46}\,5^{15}\,7^9\,11^5\,13^4\,17^3\,19^2\,23^2\,29\,31\,37\,41\,43\,47}V^{50} \\
+\; & \frac{838332469367518203826092449014743625539656513 7359287}{2^2\,3^{48}\,5^{14}\,7^{10}\,11^4\,13^4\,17^3\,19^2\,23^2\,29\,31\,37\,41\,43\,47}V^{52} \\
+\; & \frac{4159082295083000370124175392269663844075356583256276 27}{2\,3^{49}\,5^{16}\,7^9\,11^5\,13^3\,17^3\,19^2\,23^2\,29\,31\,37\,41\,43\,47\,53}V^{54} \\
+\; & \left(\frac{47076152328972740213484379897517522 2979}{2^3\,3^{53}\,5^{18}\,7^9\,11^6\,13^4}\right. \\
+\; & \left.\frac{237429602166012268 33732}{7^9\,11^6\,13^4\,17^3\,19^3\,23^2\,29^2\,31\,37\,41\,43\,47\,53}\right)V^{56} \\
+\; & \left(\frac{10325453506033593902438034196061025 4939}{2\,3^{55}\,5^{18}\,7^8\,11^6\,13^4\,17}\right. \\
+\; & \left.\frac{102051496578471262 54970}{7^9\,11^6\,13^4\,17^3\,19^3\,23^2\,29^2\,31\,37\,41\,43\,47\,53}\right)V^{58} \\
+\; & \left(\frac{1674300782436897952287940766663293 2269}{2^2\,3^{56}\,5^{15}\,7^{11}\,11^6\,13^4}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{1347525195246963877448278 2}{7^{11}\,11^{6}\,13^{5}\,17^{3}\,19^{3}\,23^{2}\,29\,31^{2}\,37\,41\,43\,47\,53\,59}\Bigg)\,V^{60}\\
&- \quad \Bigg(\frac{15438484096253401247073337405709319083010 7}{2\,3^{58}\,5^{20}\,7^{10}\,11^{6}\,13^{5}}\\
&+ \quad \frac{43565193244189659925867827329 7}{7^{11}\,11^{6}\,13^{5}\,17^{4}\,19^{3}\,23^{2}\,29^{2}\,31^{2}\,37\,41\,43\,47\,53\,59\,61}\Bigg)\,V^{62}\\
&- \quad \Bigg(\frac{45278427475446280290227647279515756210389 9}{2^{6}\,3^{61}\,5^{20}\,7^{11}\,11^{4}\,13^{4}}\\
&+ \quad \frac{57288741570071010814728876421}{7^{12}\,11^{6}\,13^{5}\,17^{4}\,19^{3}\,23^{2}\,29^{2}\,31\,37\,41\,43\,47\,53\,59\,61}\Bigg)\,V^{64}\\
&- \quad \Bigg(\frac{932747479601182448004828911670338937499460 09}{2\,3^{62}\,5^{21}\,7^{11}\,11^{6}\,13^{5}}\\
&+ \quad \frac{201745014410154961441219038 80}{7^{12}\,11^{6}\,13^{5}\,17^{4}\,19^{3}\,23^{2}\,29^{2}\,31^{2}\,37\,41\,47\,53\,59\,61}\Bigg)\,V^{66}\\
&- \quad \Bigg(\frac{25950583145072839403742846837472460457885798 29}{2^{2}\,3^{64}\,5^{21}\,7^{11}\,11^{6}\,13^{6}}\\
&+ \quad \frac{106720766466734564494409498177754 8}{7^{11}\,11^{6}\,13^{6}\,17^{4}\,19^{3}\,23^{3}\,29^{2}\,31^{2}\,37\,41\,43\,47\,53\,59\,61\,67}\Bigg)\,V^{68}\\
&- \quad \Bigg(\frac{147672807016578801159881310073597410693059851}{2\,3^{66}\,5^{21}\,7^{12}\,11^{7}\,13^{4}}\\
&+ \quad \frac{16092572920987934408676131550481 0}{7^{11}\,11^{7}\,13^{6}\,17^{3}\,19^{4}\,23^{3}\,29\,31^{2}\,37\,41\,43\,47\,53\,59\,61\,67}\Bigg)\,V^{70}\\
&- \quad \Bigg(\frac{3902157820144885924002852000114112529084571310 7}{2^{3}\,3^{67}\,5^{20}\,7^{14}\,11^{7}\,13^{3}}\\
&+ \quad \frac{92271258695512250103569373714158929929 38}{7^{14}\,11^{7}\,13^{6}\,17^{4}\,19^{4}\,23^{3}\,29^{2}\,31^{2}\,37^{2}\,41\,43\,47\,53\,59\,61\,67\,71}\Bigg)\,V^{72}\\
&- \quad \Bigg(\frac{76334728656543880537199504646100369575620917103 51}{2\,3^{70}\,5^{23}\,7^{13}\,11^{8}\,13^{3}}\\
&+ \quad \frac{389210692782618834210924249509151050094844 9}{7^{14}\,11^{8}\,13^{6}\,17^{4}\,19^{4}\,23^{3}\,29^{2}\,31^{2}\,37^{2}\,41\,43\,47\,53\,59\,61\,67\,71\,73}\Bigg)\,V^{74}\\
&- \quad \Bigg(\frac{677420444328896787949969455703133028951942798613357}{2^{2}\,3^{72}\,5^{24}\,7^{13}\,11^{7}\,13^{4}}\\
&+ \quad \frac{6921526432544325082887790298576256682 19}{7^{14}\,11^{7}\,13^{6}\,17^{4}\,19^{4}\,23^{3}\,29^{2}\,31^{2}\,37\,41\,43\,47\,53\,59\,61\,67\,71\,73}\Bigg)\,V^{76}\\
&- \quad \Bigg(\frac{96115585871180451350439504698848377028969833881 33}{2\,3^{73}\,5^{24}\,7^{13}\,11^{7}\,13^{3}}\\
&+ \quad \frac{108179377138569736997768235319216318685611}{7^{15}\,11^{8}\,13^{6}\,17^{4}\,19^{3}\,23^{3}\,29^{2}\,31^{2}\,37^{2}\,41\,43\,47\,53\,59\,61\,67\,71\,73}\Bigg)\,V^{78}\\
&- \quad \Bigg(\frac{643133108246837316426220989770158028036068089336498527}{2^{4}\,3^{75}\,5^{26}\,7^{13}\,11^{8}\,13^{4}}\\
&+ \quad \frac{24996516452151634416996415287741300576899382 74}{7^{15}\,11^{8}\,13^{5}\,17^{5}\,19^{4}\,23^{3}\,29^{2}\,31^{2}\,37^{2}\,41^{2}\,43\,47\,53\,59\,61\,67\,71\,73\,79}\Bigg)\,V^{80}
\end{aligned}
$$

$$
\begin{aligned}
&- \Bigg(\frac{588000071618029332011644884148997413609480727619230348 9}{2\,3^{80}\,5^{25}\,7^{15}\,11^{8}\,13^{3}} \\
&+ \frac{60887367115008041463686131078779737625289997452 4}{7^{16}\,11^{8}\,13^{7}\,17^{5}\,19^{4}\,23^{3}\,29^{2}\,31^{2}\,37^{2}\,41^{2}\,43\,47\,53\,59\,61\,67\,71\,73\,79} \Bigg) V^{82} \\
&- \Bigg(\frac{67950093416665836624322615109825783711}{2^{2}\,3^{81}\,5^{24}\,7^{9}} \\
&+ \frac{218640552056180107705138867139836150617384978946}{5^{26}\,7^{16}\,11^{8}\,13^{7}\,17^{5}\,19^{4}\,23^{3}\,29^{2}\,31^{2}\,37^{2}\,41\,43^{2}} \\
&+ \frac{57802760323023892}{17^{5}\,19^{4}\,23^{3}\,29^{2}\,31^{2}\,37^{2}\,41\,43^{2}\,47\,53\,59\,61\,67\,71\,73\,79\,83} \Bigg) V^{84} \\
&- \Bigg(\frac{2862915725716590838052494793698868398433}{2\,3^{83}\,5^{28}\,7^{8}} \\
&+ \frac{73527263938361840013521065984540983019815726873 73}{5^{28}\,7^{15}\,11^{9}\,13^{7}\,17^{5}\,19^{4}\,23^{4}\,29^{3}\,31^{2}\,37^{2}} \\
&+ \frac{8088284813421368027926}{17^{5}\,19^{4}\,23^{4}\,29^{2}\,31^{2}\,37^{2}\,41^{2}\,43^{2}\,47\,53\,59\,61\,67\,71\,73\,79\,83} \Bigg) V^{86} \\
&- \Bigg(\frac{16235383552717455847451674503570491952969 7}{2^{3}\,3^{85}\,5^{28}\,7^{9}} \\
&+ \frac{91134516112567617820082798393362599367045440379 03}{5^{28}\,7^{16}\,11^{8}\,13^{7}\,17^{5}\,19^{4}\,23^{4}\,29^{3}\,31^{2}\,37^{2}} \\
&+ \frac{7144828593107202894 31}{17^{5}\,19^{4}\,23^{4}\,29^{3}\,31^{2}\,37^{2}\,41^{2}\,43\,47\,53\,59\,61\,67\,71\,73\,79\,83} \Bigg) V^{88} \\
&- \Bigg(\frac{13560458268847593176946767714487307000295 3}{2\,3^{86}\,5^{27}\,7^{11}} \\
&+ \frac{53399188168342139161678355428883405484723361898 98}{5^{27}\,7^{17}\,11^{8}\,13^{7}\,17^{5}\,19^{4}\,23^{4}\,29^{3}\,31^{2}\,37^{2}} \\
&+ \frac{91700492961222210321302 38}{17^{5}\,19^{5}\,23^{4}\,29^{3}\,31^{2}\,37^{2}\,41^{2}\,43^{2}\,47\,53\,59\,61\,67\,71\,73\,79\,83\,89} \Bigg) V^{90} \\
&- \Bigg(\frac{47192632231777770888333145260955643812808 3}{2^{2}\,3^{89}\,5^{29}\,7^{10}} \\
&+ \frac{238877778424907093269907902660448645890101129313814843}{5^{29}\,7^{17}\,11^{10}\,13^{8}\,17^{5}\,19^{5}\,23^{4}\,29^{3}\,31^{3}\,37} \\
&+ \frac{17355407260625422217891929362 0}{17^{6}\,19^{5}\,23^{4}\,29^{3}\,31^{3}\,37^{2}\,41^{2}\,43^{2}\,47^{2}\,53\,59\,61\,67\,71\,73\,79\,83\,89} \Bigg) V^{92} \\
&- \Bigg(\frac{37974361997658175543170336409293856688091 73}{2\,3^{91}\,5^{27}\,7^{11}} \\
&+ \frac{620311833099552408010286737804921639220433583185388 26}{5^{29}\,7^{17}\,11^{10}\,13^{8}\,17^{6}\,19^{5}\,23^{3}\,29^{3}\,31^{3}\,37^{2}} \\
&+ \frac{8273835042008641754442575 18}{17^{6}\,19^{5}\,23^{3}\,29^{3}\,31^{3}\,37^{2}\,41^{2}\,43^{2}\,47^{2}\,53\,59\,61\,67\,71\,73\,79\,83\,89} \Bigg) V^{94} \\
&- \Bigg(\frac{2648046807874727945014969125635126106835925 11}{2^{5}\,3^{92}\,5^{29}\,7^{13}} \\
&+ \frac{858447609893232567223557068482982345555677027}{5^{30}\,7^{17}\,11^{10}\,13^{8}\,17^{6}\,19^{5}\,23^{4}\,29^{3}}
\end{aligned}
$$

$$+ \quad \frac{15866921185581024582777185}{13^8\,17^6\,19^5\,23^4\,29^3\,31^3\,37^2\,41^2\,43^2\,47\,53\,59\,61\,67\,73}$$
$$\left. + \quad \frac{126330}{29^3\,31^3\,37^2\,41^2\,43^2\,47\,59\,61\,67\,71\,73\,79\,83\,89}\right) V^{96} \;+\; O\left(x^{98}\right).$$

To handle these computations we apply the integral representation (3), i. e. we invert the diffeomorphism $V(t)$ by means of series and integrate the result with respect to V. This could be easily done by the following Maple program.

```
> Vt := t*(1-(coth(t)-1/t)^2):
> Order := 97:
> int(solve(series(Vt, t) = V, t), V);
```

The coefficients presented here are slighlty reformatted to fit the page. ■

Following lemmata prepare us to prove that $L(V)$ is bounded by its Taylor polynomial of degree 49. We start with the following upper bound for $V(t)$.

Lemma 2. *For $t > 2$ the following inequality holds*

$$V(t) \leq 2 - 1/t.$$

Proof. This inequality is equivalent to the following one

$$(\coth(t) - 1)\,(t\,\coth(t) + t - 2) > 0,$$

which is obvious when $t > 2$. ■

The following lemma states that function $L(V)$ is bounded by its Taylor polynomial of degree 49. We present a computational proof, which is not specific for this problem. This method can be easily utilized to claim any type of inequalities in one variable.

Lemma 3. *The following inequality holds for V from* $[\,0;\,2\,[$

$$L(V) \geq T_{49}(V),$$

where $T_{49}(V)$ is the Taylor polynomial of order 49 *constructed from $L(V)$ in a neighborhood of* 0.

Proof. Let us start claiming that

$$T_{49}(V) = T_{48}(V) = \sum_{i=1}^{24} t_{2i}\,V^{2i},$$

which is obvious consequence of Lemma 1.

We need to prove $R_{48}(V) = L(V) - T_{48}(V) \geq 0$ for all possible V. It is sufficient to prove an inequality

$$R_{48}(t) = R_{48}(V(t)) \geq 0 \tag{4}$$

for all t. For $t \leq 0.8$ we state the inequality (4) estimating a derivative of $R_{48}(t)$ by means of interval arithmetic. The derivative has a zero of order 49 when $t = 0$. To avoid computational

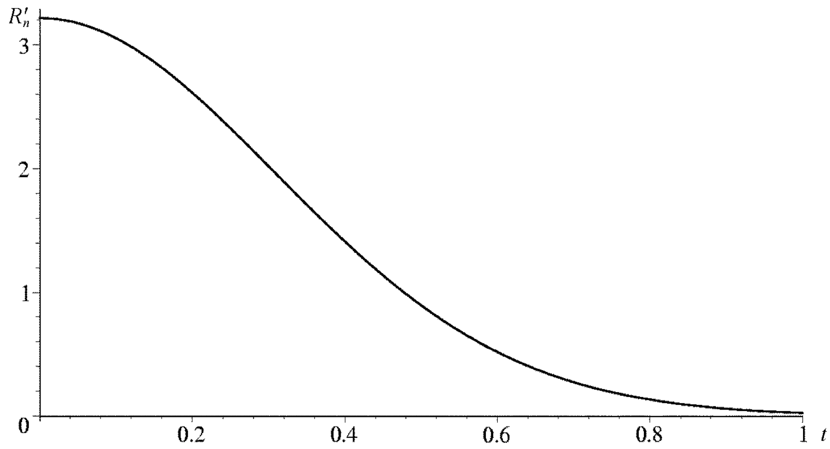

Fig. 1. Behavior of the function $R'_n(t) \times 10^{15}$ in a zero neighborhood.

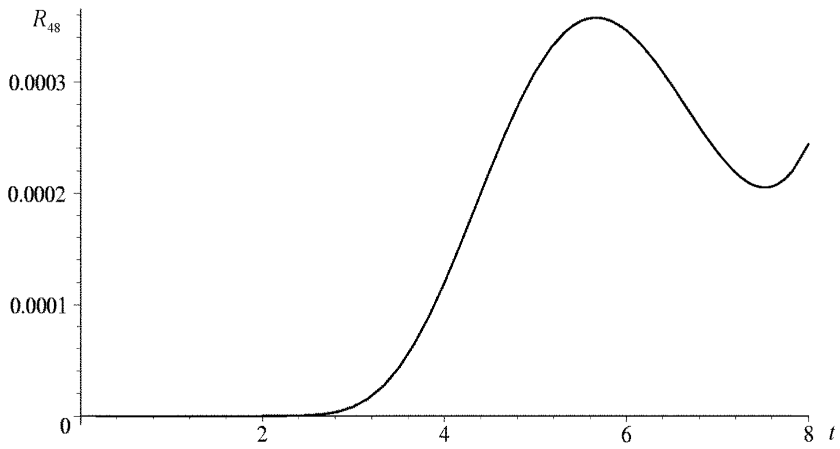

Fig. 2. Graph of the function $R_{48}(t)$ on $[\,0;\,8\,]$.

difficulties we normalize the derivative in zero with a multiplier t^{-49} and denote the normalized derivative as $R'_n(t)$:

$$R'_n(t) = \frac{1}{t^{49}}\,\frac{dR_{48}(V)}{dV}\,(t).$$

We depict a graph of the function in Fig. 1. The function $R'_n(t)$ has a positive limit

$$\begin{aligned}\lim_{t\to 0} R'_n(t) \;&=\; \frac{15312621927355180839940}{7^9\, 11^5\, 13^4\, 17^3\, 19^2\, 23^2\, 29\, 31\, 37\, 41\, 43\, 47} + \\ &+\; \frac{24243923906662864859970002659}{3^{46}\, 5^{13}\, 7^5\, 11^5\, 13^4}\end{aligned}$$

in zero, so we are able to compute values of $R'_n(t)$ in a zero neighborhood and put them in a table. This table shows that the derivative is positive for small t. It is easy to see that $R_{48}(0) = 0$, hence $R_{48}(t)$ is positive in the neighborhood also.

The segment $[\,0.8;\,8.0\,]$ could be splitted approximately into 17000 interval numbers which validate the inequality $R_{48}(t) > 0$. There is no room here for all the computations. We only present a graph of $R_{48}(t)$ on Fig. 2.

To complete the proof we need to show that $R_{48}(t)$ is positive for $t > 8$. Because $R_{48}(8)$ is

Values of function $R'_n(t)$ for t from $[0; 0.8]$

t	$R'_n(t) \times 10^{15}$	t	$R'_n(t) \times 10^{15}$
[0.000..0.400]	[0.532..3.220]	[0.400..0.500]	[0.134..1.966]
[0.500..0.550]	[0.132..1.388]	[0.550..0.575]	[0.260..1.009]
[0.575..0.600]	[0.124..0.972]	[0.600..0.613]	[0.252..0.739]
[0.613..0.626]	[0.196..0.719]	[0.626..0.639]	[0.141..0.703]
[0.639..0.652]	[0.086..0.691]	[0.652..0.665]	[0.030..0.684]
[0.665..0.672]	[0.150..0.524]	[0.672..0.679]	[0.126..0.517]
[0.679..0.686]	[0.103..0.511]	[0.686..0.693]	[0.079..0.506]
[0.693..0.700]	[0.056..0.502]	[0.700..0.707]	[0.033..0.500]
[0.707..0.714]	[0.009..0.498]	[0.714..0.718]	[0.100..0.390]
[0.718..0.722]	[0.089..0.387]	[0.722..0.726]	[0.078..0.385]
[0.726..0.730]	[0.067..0.383]	[0.730..0.734]	[0.057..0.381]
[0.734..0.738]	[0.046..0.379]	[0.738..0.742]	[0.035..0.378]
[0.742..0.746]	[0.025..0.377]	[0.746..0.750]	[0.014..0.377]
[0.750..0.754]	[0.003..0.376]	[0.754..0.756]	[0.090..0.282]
[0.756..0.758]	[0.086..0.280]	[0.758..0.760]	[0.082..0.279]
[0.760..0.762]	[0.078..0.278]	[0.762..0.764]	[0.074..0.277]
[0.764..0.766]	[0.070..0.276]	[0.766..0.768]	[0.066..0.275]
[0.768..0.770]	[0.062..0.274]	[0.770..0.772]	[0.058..0.273]
[0.772..0.774]	[0.054..0.272]	[0.774..0.776]	[0.049..0.272]
[0.776..0.778]	[0.045..0.271]	[0.778..0.780]	[0.041..0.270]
[0.780..0.782]	[0.037..0.270]	[0.782..0.784]	[0.033..0.269]
[0.784..0.786]	[0.029..0.269]	[0.786..0.788]	[0.025..0.269]
[0.788..0.790]	[0.021..0.268]	[0.790..0.792]	[0.017..0.268]
[0.792..0.794]	[0.013..0.268]	[0.794..0.796]	[0.008..0.268]
[0.796..0.798]	[0.004..0.268]	[0.798..0.800]	[0.000..0.268]

positive it is sufficient to prove that derivative

$$R'_v(t) = \frac{dR_{48}(V(t))}{dV}\,(t) = t - \sum_{\nu=0}^{48} (\nu+1)\,t_\nu\,V(t)^\nu$$

is positive for $t > 8$. Moreover, if we substitute $V(t)$ with the upper bound from Lemma 2 then we get a stronger inequality

$$t - \sum_{\nu=0}^{48} (\nu+1)\,t_\nu\,(2-1/t)^\nu > 0.$$

We can multiply it by t^{47} and rewrite it as a polynomial inequality $P(t-8) > 0$. This inequaliy is obviously valid for $t > 8$ because P has all coefficients greater then zero. This finishes the proof. ■

References

[1] Pinsker M.S. Information and information stability of random variables and processes. Moskva: Izv. Akad. Nauk, 1960. in Russian.

[2] Csiszár I. Information-type measures of difference of probability distributions and indirect observations // Studia Sci. Math. Hungar. 1967. Vol. 2. P. 229–318.

[3] Kemperman J.H.B. On the optimum rate of transmitting information. B.: Springer-Verlag: Lecture Notes in Mathematics, 1967. P. 126–169.

[4] Kemperman J.H.B. On the optimum rate of transmitting information // Ann. Math. Statist., Dec. 1969. Vol. 40. P. 2156–2177.

[5] Kullback S. A lower bound for discrimination information in terms of variation // IEEE Trans. Inform. Theory. Jan. 1967. Vol. 13. P. 126–127.

[6] Kullback S. Correction to “a lower bound for discrimination information in terms of variation” // IEEE Trans. Inform. Theory. 1970. Vol. 16. P. 652.

[7] Krafft O., Schmitz N. A note on Hoeffding inequality // J. Amer. Statist. Assoc. 1969. Vol. 64. P. 907–912.

[8] Toussaint G.T. Sharper lower bounds for discrimination in terms of variation // IEEE Trans. Inform. Theory. Nov. 1975. Vol. 21. P. 99–100.

[9] Topsøe F. Bounds for entropy and divergence of distributions over two-elment set // J. Ineq. Pure Appl. Math. 2001. http://jipam.vu.edu.au/v2n2/044.00.html.

[10] Fedotov A., Harremoës P., Topsøe F. Refinements of Pinsker Inequality // IEEE Trans. Inform. Theory. June 2003. Vol. 49. P. 1491–1498.

[11] Vajda I. Note on discrimination information and variation // IEEE Trans. Inform. Theory. Nov. 1970. P. 771–773.

Received for publication June 20, 2003