ON THE GENERALIZED HEAT KERNEL*

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В данной работе мы исследуем уравнение

$$\frac{\partial}{\partial t}u(x,t) = -c^2(-\triangle)^k u(x,t)$$

с начальными условиями

$$u(x,0) = f(x),$$

где $x \in \mathbb{R}^n$, $\mathbb{R}^n$ — n-мерное евклидово пространство. Оператор $\triangle^k$ называется оператором Лапласа, итерированным k раз, и определяется как

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^k,$$

где n — размерность евклидова пространства $\mathbb{R}^n$; $u(x,t)$ — неизвестная функция от $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times [0,\infty)$;, $f(x)$ — заданная обобщенная функция; k — неотрицательное целое число; c — положительная постоянная.

Решение такого уравнения, называемое обобщенным ядром уравнения теплопроводности, имеет интересные свойства и связано с решением уравнения теплопроводности.

Introduction

It is well known that for the heat equation

$$\frac{\partial}{\partial t}u(x,t) = c^2\triangle u(x,t) \tag{0.1}$$

with the initial condition

$$u(x,0) = f(x),$$

where $\triangle = \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times [0,\infty)$, we obtain the solution

$$u(x,t) = \frac{1}{(4c^2\pi t)^{n/2}}\int_{\mathbb{R}^n} \exp\left[-\frac{|x-y|^2}{4c^2t}\right] f(y)dy.$$

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Alternatevely, this solution can be represented in the convolution form

$$u(x,t) = E(x,t) * f(x), \tag{0.2}$$

where

$$E(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \exp\left[-\frac{|x|^2}{4c^2 t}\right]. \tag{0.3}$$

The function (0.3) called *the heat kernel*, where $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ and $t > 0$, see [1, p. 208, 209].

Moreover, we obtain $E(x,t) \to \delta$ as $t \to 0$, where δ is the Dirac-delta function. We can extend (0.1) to the equation

$$\frac{\partial}{\partial t} u(x,t) = -c^2 \triangle^2 u(x,t) \tag{0.4}$$

with the initial condition

$$u(x,0) = f(x),$$

where $\triangle^2 = \triangle\triangle$ is the biharmonic operator, that is

$$\triangle^2 = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^2.$$

Using the n-dimensional Fourier transform we can find the following solution of (0.4)

$$u(x,t) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} e^{-c^2|\xi|^4 t + i(\xi, x-y)} f(y)\, dy\, d\xi. \tag{0.5}$$

Using (0.5) $u(x,t)$ can be rewritten in the convolution form

$$u(x,t) = E(x,t) * f(x),$$

where

$$E(x,t) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} e^{-c^2|\xi|^4 t + i(\xi,x)}\, d\xi, \tag{0.6}$$

$|\xi|^4 = (\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2)^2$ and $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$. The function $E(x,t)$ in (0.6) is the kernel of (0.4), $E(x,t) \to \delta$ as $t \to 0$ since

$$\lim_{t\to 0} E(x,t) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} e^{(\xi,x)i}, d\xi = \delta,$$

see [3, p. 396, Eq. (10.2.19(b))].

Now, the purpose of this work is to study the equation

$$\frac{\partial}{\partial t} u(x,t) = -c^2(-\triangle)^k u(x,t) \tag{0.7}$$

with the initial condition

$$u(x,0) = f(x), \text{ for } x \in \mathbb{R}^n,$$

where the operator $\triangle^k$ denotes *the Laplace operator iterated k-times.* This operator is defined as follows

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^k, \tag{0.8}$$

where n is the dimension of Euclidean space $\mathbb{R}^n$, $u(x,t)$ is an unknown function, $(x,t) = (x_1, x_2, \dots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, $f(x)$ is the given generalized function, k is a nonnegative integer and c is a positive constant.

We obtain $u(x,t) = E(x,t) * f(x)$ as a solution of (0.7), where

$$E(x,t) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \exp\left[-c^2 \left(\sum_{i=1}^{p} \xi_i^2\right)^k t + i(\xi, x)\right] d\xi. \tag{0.9}$$

All properties of $E(x,t)$ in (0.9) will be studied in details.

Now, if we set $k = 1$ in (0.9) then (0.9) reduces to (0.3), which is the kernel of (0.1). Also, if we set $k = 2$ in (0.9), then (0.9) reduces to (0.6), which is the kernel of (0.4).

1. Preliminaries

Definition 1.1. *Let $f(x) \in \mathbb{L}_1(\mathbb{R}^n)$ be the space of integrable functions in $\mathbb{R}^n$. The Fourier transform of $f(x)$ is defined by*

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-i(\xi,x)} f(x)\, dx, \tag{1.1}$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_n)$, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ is the usual inner product in $\mathbb{R}^n$, $dx = dx_1\, dx_2 \dots dx_n$.

The inverse Fourier transform is given by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{f}(\xi)\, d\xi. \tag{1.2}$$

Lemma 1.1. *Given the function*

$$f(x) = \exp\left[-\left(\sum_{i=1}^{n} x_i^2\right)^k\right],$$

where $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Then

$$\left|\int\limits_{\mathbb{R}^n} f(x)\, dx\right| \le \frac{\pi^{\frac{n}{2}}}{k} \frac{\Gamma\left(\frac{n}{2k}\right)}{\Gamma\left(\frac{n}{2}\right)}, \tag{1.3}$$

where Γ denotes the Gamma function. Therefore, $\int\limits_{\mathbb{R}^n} f(x)\, dx$ is bounded.

Proof. We have

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_{\mathbb{R}^n} \exp\left[-\left(\sum_{i=1}^{p} x_i^2\right)^k\right] dx.$$

Let us transform to bipolar coordinates

$$x_1 = r\omega_1, x_2 = r\omega_2, \ldots, x_n = r\omega_n,$$

where $\sum_{i=1}^{n} \omega_i^2 = 1$.

Thus

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_{\mathbb{R}^n} e^{-r^{2k}} r^{n-1}\,dr\,d\Omega_n,$$

where

$$dx = r^{n-1}\,dr\,d\Omega_n, \tag{1.4}$$

$d\Omega_n$ is the element of surface area on the unit sphere in $\mathbb{R}^n$. By direct computation we obtain

$$\int_{\mathbb{R}^n} f(x)\,dx = \Omega_n \int_0^\infty e^{-r^{2k}} r^{n-1}\,dr, \tag{1.5}$$

where $\Omega_n = \dfrac{2\pi^{n/2}}{\Gamma(n/2)}$.

When $u = r^{2k}$, we then obtain

$$\left|\int_{\mathbb{R}^n} f(x)\,dx\right| \le \frac{\Omega_n}{2k}\int_0^\infty e^{-u} u^{\frac{n}{2k}-1}\,du = \frac{\Omega_n}{2k}\Gamma\left(\frac{n}{2k}\right) = \frac{\pi^{\frac{n}{2}}}{k}\frac{\Gamma\left(\frac{n}{2k}\right)}{\Gamma\left(\frac{n}{2}\right)}. \tag{1.6}$$

Therefore, $\int_{\mathbb{R}^n} f(x)\,dx$ is bounded. □

Lemma 1.2. *For all $t > 0$ and all $x \in \mathbb{R}$ we have*

$$\int_{-\infty}^{\infty} \exp\left(-c^2\xi^2 t\right)\,d\xi = \sqrt{\frac{\pi}{c^2 t}} \tag{1.7}$$

and

$$\int_{-\infty}^{\infty} \exp\left[-c^2\xi^2 t + i\xi x\right]\,d\xi = \sqrt{\frac{\pi}{c^2 t}}\exp\left(-\frac{x^2}{4c^2 t}\right), \tag{1.8}$$

where c is a positive constant.

Proof. See [2, p. 117, 118]. □

2. Main Results

Theorem 2.1. *Given the equation*

$$\frac{\partial}{\partial t} u(x,t) = -c^2(-\triangle)^k u(x,t) \tag{2.1}$$

with the initial condition

$$u(x,0) = f(x), \tag{2.2}$$

where $\triangle^k$ is the Laplace operator iterated k-times defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^k,$$

where n is the dimension of Euclidean space $\mathbb{R}^n$, k is a nonnegative integer, $u(x,t)$ is an unknown function, $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, $f(x)$ is the given generalized function, and c is a positive constant. Then we obtain that

$$u(x,t) = E(x,t) * f(x) \tag{2.3}$$

is a solution of (2.1), which satisfies (2.2) where $E(x,t)$ is the kernel of (2.1) defined by

$$E(x,t) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \exp\left[-c^2\left(\sum_{i=1}^n \xi_i^2\right)^k t + i(\xi,x)\right] d\xi. \tag{2.4}$$

Proof. Applying the Fourier transform (1.1) to both sides of (2.1), we obtain

$$\frac{\partial}{\partial t}\widehat{u}(\xi,t) = -c^2\left(\sum_{i=1}^n \xi_i^2\right)^k \widehat{u}(\xi,t).$$

Thus,

$$\widehat{u}(\xi,t) = K(\xi)\exp\left[-c^2\left(\sum_{i=1}^n \xi_i^2\right)^k t\right], \tag{2.5}$$

where $K(\xi)$ is a constant and $\widehat{u}(\xi,0) = K(\xi)$.

$\widehat{u}(\xi,t)$ in (2.5) is bounded and from (2.2) we have

$$K(\xi) = \widehat{u}(\xi,0) = \widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-i(\xi,x)} f(x)\,dx \tag{2.6}$$

and using the inversion in (1.2) we obtain from (2.5) and (2.6)

$$\begin{aligned} u(x,t) &= \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{i(\xi,x)}\widehat{u}(\xi,t)\,d\xi = \\ &= \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{i(\xi,x)} e^{-i(\xi,y)} f(y) \exp\left[-c^2\left(\sum_{i=1}^n \xi_i^2\right)^k t\right] dy\,d\xi. \end{aligned}$$

Therefore,

$$u(x,t) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} e^{i(\xi, x-y)} \exp\left[-c^2 \left(\sum_{i=1}^n \xi_i^2\right)^k t\right] f(y)\, dy\, d\xi \tag{2.7}$$

or

$$u(x,t) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \exp\left[-c^2 \left(\sum_{i=1}^n \xi_i^2\right)^k t + i(\xi, x-y)\right] f(y)\, dy\, d\xi. \tag{2.8}$$

Set

$$E(x,t) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \exp\left[-c^2 \left(\sum_{i=1}^n \xi_i^2\right)^k t + i(\xi, x)\right] d\xi. \tag{2.9}$$

Thus, (2.8) can be rewritten in the convolution form

$$u(x,t) = E(x,t) * f(x), \tag{2.10}$$

where $u(x,t)$ in (2.8) is a solution of (2.1) and $E(x,t)$ is defined by (2.9). It is clear that the kernel $E(x,t)$ exists.

Moreover, since $E(x,t)$ exists, then

$$\lim_{t \to 0} E(x,t) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} e^{i(\xi, x)}\, d\xi = \delta(x), \text{ for } x \in \mathbb{R}^n. \tag{2.11}$$

See [3, p. 396, Eq. (10.2.19(b))].

From (2.11) we obtain

$$u(x,0) = \lim_{t \to 0} u(x,t) = \lim_{t \to 0} (E(x,t) * f(x)) = \delta * f(x) = f(x).$$

Thus, $u(x,t)$ in (2.3) satisfies (2.2).

In particular, if we set $k = 1$ in (2.9), then we obtain

$$\begin{aligned}
E(x,t) &= \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \exp\left[-c^2 \left(\sum_{j=1}^n \xi_j^2\right) t + i(\xi, x)\right] d\xi = \\
&= \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \exp\left[-c^2 \sum_{j=1}^n \xi_j^2 t + i \sum_{j=1}^n \xi_j x_j\right] d\xi = \\
&= \frac{1}{(2\pi)^n} \prod_{j=1}^n \int\limits_{-\infty}^{\infty} \exp\left[-c^2 \xi_j^2 t + i \xi_j x_j\right] d\xi_j = \\
&= \frac{1}{(2\pi)^n} \prod_{j=1}^n \sqrt{\frac{\pi}{c^2 t}} \exp\left(-\frac{x_j^2}{4c^2 t}\right)
\end{aligned}$$

from (1.8). Thus,

$$E(x,t) = \frac{1}{(4c^2 \pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4c^2 t}\right),$$

since

$$\left(\frac{\pi}{c^2 t}\right)^{\frac{n}{2}} \exp\left(-\frac{|x|^2}{4c^2 t}\right) = \prod_{j=1}^{n} \sqrt{\frac{\pi}{c^2 t}} \exp\left(-\frac{x_j^2}{4c^2 t}\right)$$

and $|x|^2 = \sum\limits_{i=1}^{n} x_i^2$.

Therefore, if we set $k = 1$ in (2.1) and (2.9), then (2.1) and (2.9) will be reduced to (0.1) and (0.3), respectively. If we set $k = 2$ in (2.9), then we obtain

$$E(x,t) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \exp\left[-c^2 \left(\sum_{i=1}^{n} \xi_i^2\right)^2 t + i(\xi, x)\right] d\xi =$$

$$= \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} e^{-c^2 |\xi|^4 t + i(\xi,x)}\, d\xi,$$

where $|\xi|^4 = (\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2)^2$.

Therefore, if we set $k = 2$ in (2.1) and (2.9), then (2.1) and (2.9) will be reduced to (0.4) and (0.6), respectively. □

Theorem 2.2. *The kernel $E(x,t)$ defined by (2.9) has the following properties:*

1) $E(x,t) \in C^\infty$, where C^∞ is the space of continuous infinitely differentiable functions, $x \in \mathbb{R}^n$, $t > 0$;

2) $\left(\frac{\partial}{\partial t} + c^2(-\triangle)^k\right) E(x,t) = 0$ for $t > 0$;

3) $E(x,t) > 0$ for $t > 0$;

4)

$$|E(x,t)| \le \frac{1}{2^n \pi^{n/2} k (c^2 t)^{n/2k}} \frac{\Gamma\left(\frac{n}{2k}\right)}{\Gamma\left(\frac{n}{2}\right)}, \text{ for } t > 0,$$

where Γ denotes the Gamma function. Thus $E(x,t)$ is bounded for any fixed t;

5) $\lim\limits_{t \to 0} E(x,t) = \delta$.

Proof.

1. This property follows from (2.9), since

$$\frac{\partial^n}{\partial x^n} E(x,t) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \frac{\partial^n}{\partial x^n} \exp\left[-c^2 \left(\sum_{i=1}^{n} \xi_i^2\right)^k t + i(\xi, x)\right] d\xi.$$

Thus, $E(x,t) \in C^\infty$ for $x \in \mathbb{R}^n$, $t > 0$.

2. By direct computation we obtain

$$\left(\frac{\partial}{\partial t} + c^2(-\triangle)^k\right) E(x,t) = 0$$

for $t > 0$, where $E(x,t)$ is defined by (2.9).

3. $E(x,t) > 0$ for $t > 0$ is obvious from (2.9).

4. From (2.9) we have

$$E(x,t) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \exp\left[-c^2 \left(\sum_{i=1}^{n} \xi_i^2\right)^k t + i(\xi, x)\right] d\xi.$$

Therefore,

$$|E(x,t)| \leq \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \exp\left[-c^2 t \left(\sum_{i=1}^{n} x_i^2\right)^k\right] dy.$$

Using the same procedure as in Lemma 1.1, we obtain

$$|E(x,t)| \leq \frac{1}{2^n \pi^{n/2} k (c^2 t)^{n/2k}} \frac{\Gamma\left(\frac{n}{2k}\right)}{\Gamma\left(\frac{n}{2}\right)}.$$

Thus, $E(x,t)$ is bounded for any fixed t.

5. This property is obvious from (2.11). □

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