A CLASS OF OPTIMAL CENTERED FORM INTERVAL EXTENSION*

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Для дважды дифференцируемых функций представлено простое выражение для оптимальных центрированных форм интервального расширения функций. Для всех возможных центров в пределах заданного интервала предлагаемая центрированная форма предоставляет наибольшую нижнюю и наименьшую верхнюю границы интервального расширения. Кроме того, обобщен и улучшен результат [1] и доказано, что предлагаемая центрированная форма имеет лучшую точность, чем представленная в [1].

Introduction

In applications of interval mathematics, it is extremely important to determine sharp enclosure for the range of a given function $f : D \subseteq R^n \to R^1$ on a given interval in R^n. If $f \in C^1(D)$, then, as it is well known [1, 2], an interval extension of f can be obtained using the mean value theorem, which itself has been used quite successfully in several applications [2]. The interval extension $F : I(D) \times D \to I(R)$ of f defined by

$$(IE1) \quad F(X,c) = f(c) + (X-c)^T F'(X) \tag{0.1}$$

is called a mean value form interval extension of f on X with the center $c \in X$, where $F'(X) = (F_1'(X), \ldots, F_n'(X))^T$, $F_j'(X)$ is an interval extension of $\partial f(x)/\partial x_j$ on X.

But we now consider its dependence on the point c. Baumann (1986) provided formulas for $c \in X$ such that we can obtain optimum lower or upper bounds for $f(X)$ in a sense that is made precise below. Baumann's results which were derived in a more general setting are as follows. Let $F'(X) = [\underline{l}, \bar{l}]$ and the vectors $\underline{c}$ and $\bar{c}$ are defined as follows

$$\underline{c}_i = \begin{cases} \bar{x}_i, & \bar{l}_i \le 0, \\ \underline{x}_i, & \underline{l}_i \ge 0, \\ (\bar{l}_i\underline{x}_i - \underline{l}_i\bar{x}_i)/(\bar{l}_i - \underline{l}_i), & \text{otherwise}, \end{cases} \tag{0.2}$$

$$\bar{c}_i = \begin{cases} \underline{x}_i, & \bar{l}_i \le 0, \\ \bar{x}_i, & \underline{l}_i \ge 0, \\ (\underline{l}_i\underline{x}_i - \bar{l}_i\bar{x}_i)/(\underline{l}_i - \bar{l}_i), & \text{otherwise}, \end{cases} \tag{0.3}$$

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for $i = 1, 2, \dots, n$. Then

$$\inf F(X, c) \leq \inf F(X, \underline{c}), \quad \sup F(X, \bar{c}) \leq \sup F(X, c)$$

for all $c \in X$. These formulas mean that we may use $\underline{c}$ or $\bar{c}$ instead of any other $c \in X$ when we are interested in optimum lower or optimum upper bounds of f in X, respectively, where $F'(X)$ is assumed to be fixed.

A form that yields the greatest lower bound and the lowest upper bound of a mean value form on a given interval will be called an optimum form with respect to the lower bound and the upper bound. It is obvious that the optimum form improve the efficiency of algorithms that are based on the upper or lower bound.

The purpose of this paper is, under 2 times differentiability condition, to present a class of new interval extension (optimal centered form interval extension) by combining choosing optimum c with optimum $F'(X)$, to improve and expand Baumann's results and range of application, and to prove the interval extension having smaller excess-width than Baumann's results [1].

1. Optimal centered form interval extension

For $f : D \subseteq R^n \to R^1$ and $f \in C^2(D)$, we present two classes of interval extension forms on a given interval D. We construct the following interval functions.

The first interval extension form is as follows:

$$(IE2) \quad F(X) = f(c) + (X - c)^T B \tag{1.1}$$

where $c \in X$, B is defined as follows

$$B_i = \begin{cases} [\underline{F}'_i(a_1, \dots, a_{i-1}, \underline{a}_i, a_{i+1}, \dots, a_n), \bar{F}'_i(a_1, \dots, a_{i-1}, \bar{a}_i, a_{i+1}, \dots, a_n)], \\ \underline{F}''_{ii}(X) \geq 0, 0 \in \operatorname{int}(F'_i(X)), \\ [\underline{F}'_i(a_1, \dots, a_{i-1}, \bar{a}_i, a_{i+1}, \dots, a_n), \bar{F}'_i(a_1, \dots, a_{i-1}, \underline{a}_i, a_{i+1}, \dots, a_n)], \\ \underline{F}''_{ii}(X) \leq 0, 0 \in \operatorname{int}(F'_i(X)), \\ F'_i(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n), \quad \text{otherwise}, \end{cases} \tag{1.2}$$

where

$$a_i = \begin{cases} \underline{X}_i, & (\underline{F}_i''(X) \geq 0, \bar{F}'_i(X) \leq 0) \text{ or } (\bar{F}''_{ii}(X) \leq 0, \underline{F}'_i(X) \geq 0), \\ \bar{X}_i, & (\underline{F}''_i(X) \geq 0, \underline{F}'_i(X) \geq 0) \text{ or } (\bar{F}''_{ii}(X) \leq 0, \bar{F}'_i(X) \leq 0) \quad (i = 1, 2, \dots, n), \\ X_i, & \text{otherwise}, \end{cases}$$

and $F''_{ii}(X)$ is an interval extension of $\partial^2 f(x)/\partial x_i^2$ on X.

The other interval extension is

$$(IE3) \quad F_0(X, c) = [\inf(f(\underline{c}) + (X - \underline{c})^T B), \ \sup(f(\bar{c}) + (X - \bar{c})^T B)] \tag{1.3}$$

where the value of the vector B is the same as (1.2), the vectors $\underline{c}$, $\bar{c}$ are defined as follows

$$\underline{c}_i = \begin{cases} \bar{x}_i, & \bar{B}_i \leq 0, \\ \underline{x}_i, & \underline{B}_i \geq 0, \\ (\bar{B}_i \underline{x}_i - \underline{B}_i \bar{x}_i)/(\bar{B}_i - \underline{B}_i), & \text{otherwise} \end{cases} \quad (i = 1, 2, \dots, n), \tag{1.4}$$

$$\bar{c}_i = \begin{cases} \underline{x}_i, & \bar{B}_i \leq 0, \\ \bar{x}_i, & \underline{B}_i \geq 0, \\ (\underline{B}_i \underline{x}_i - \bar{B}_i \bar{x}_i)/(\underline{B}_i - \bar{B}_i), & \text{otherwise} \end{cases} \quad (i = 1, 2, \dots, n). \tag{1.5}$$

2. Main theory

Theorem 2.1. *Assume that $f : D \subseteq R^n \to R^1$ and $f \in C^2(D)$. Then the interval function $(IE2)$ $F(X)$ defined by (1.1) is an interval extension of $f(x)$, and satisfies*

$$(IE2) \quad F(X) \subseteq (IE1)F(X, c). \tag{2.1}$$

Proof. Firstly, $\forall x \in X$, for $(IE2)F(X)$, it is obvious that

$$F(x) = f(x). \tag{2.2}$$

Secondly, $\forall x \in X$, let the range of $f'(x)$ on X be $f'(X)$ then $\forall c \in X$, one has

$$f(x) \in f(c) + (X - c)^T \ f'(X) = f(c) + \sum_{i=1}^{n} (X_i - c_i) f_i'(X), \tag{2.3}$$

$$f(x) \in f(c) + (X - c)^T \ f'(X) \subseteq f(c) + (X - c)^T F'(X). \tag{2.4}$$

Now discuss respectively as follows:

(1) when $\underline{F}_{ii}''(X) \geq 0$ $(i = 1, \ldots, n)$ the following three cases are discussed respectively.

If $\underline{F_i'}(X) \geq 0$, then

choose $a_i = \bar{X}_i \subseteq X_i$, that is, $B_i = F_i'(a_1, \ldots, a_{i-1}, \bar{X}_i, a_{i+1}, \ldots, a_n)$, one has

$$\sup(f_i'(X)) \leq \sup(B_i) \leq \sup(F_i'(X))$$

for the interval $(X_i - c_i)B_i$, one has

$$\sup((X_i - c_i)B_i) = \max\left\{(\bar{X}_i - c_i)\bar{B}_i, (\bar{X}_i - c_i)\underline{B}_i, (\underline{X}_i - c_i)\bar{B}_i, (\underline{X}_i - c_i)\underline{B}_i\right\} =$$

$$= (\bar{X}_i - c_i)\bar{B}_i \geq (\bar{X}_i - c_i)\sup(f_i'(X)) \geq \sup((X_i - c_i)f_i'(X)),$$

$$\inf((X_i - c_i)B_i) = \min\left\{(\bar{X}_i - c_i)\bar{B}_i, (\bar{X}_i - c_i)\underline{B}_i, (\underline{X}_i - c_i)\bar{B}_i, (\underline{X}_i - c_i)\underline{B}_i\right\} =$$

$$= (\underline{X}_i - c_i)\bar{B}_i \leq (\underline{X}_i - c_i)\sup(f_i'(X)) \leq \inf((X_i - c_i)f_i'(X)),$$

and, from the above remarks, one has $(X_i - c_i)f_i'(X) \subseteq (X_i - c_i)B_i$.

It can be derived analogously that $(X_i - c_i)B_i \subseteq (X_i - c_i)F_i'(X)$. Therefore, one has

$$(X_i - c_i)f_i'(X) \subseteq (X_i - c_i)B_i \subseteq (X_i - c_i)F_i'(X).$$

With the same reason, it follows that when $\bar{F}_i'(X) \leq 0$ and $0 \in \text{int}(F_i'(X))$, we have

$$(X_i - c_i)f_i'(X) \subseteq (X_i - c_i)B_i \subseteq (X_i - c_i)F_i'(X).$$

From the above discussion, it follows that for $\underline{F}_{ii}''(X) \geq 0$ $(i = 1, \ldots, n)$, one has

$$(X_i - c_i)f_i'(X) \subseteq (X_i - c_i)B_i \subseteq (X_i - c_i)F_i'(X).$$

And with the same reason, it follows that

(2) under the case of $\bar{F}_i''(X) \leq 0$ $(i = 1, \ldots, n)$ and $0 \in \text{int}(F_i''(X))$, one has

$$(X_i - c_i)f_i'(X) \subseteq (X_i - c_i)B_i \subseteq (X_i - c_i)F_i'(X).$$

Combining (1), (2) with (2.3), $f(x) \in (IE2)F(X)$ can be obtained. It follows that interval function $(IE2)$ $F(X)$ is interval extension of $f(x)$.

From $(X_i - c_i)f_i'(X) \subseteq (X_i - c_i)B_i \subseteq (X_i - c_i)F_i'(X)(i = 1, \ldots, n)$ and (2.3), (2.4), (2.1) can be obtained, which completes the proof. □

Theorem 2.2. *Assume that* $f : D \subseteq R^n \to R^1$ *and* $f \in C^2(D)$. *Then the interval function* $(IE3)F_0(X, c)$ *defined by (1.3) is an interval extension of* $f(x)$, *and satisfies*

$$(IE3) \quad F_0(X, c) \subseteq (IE2)F(X) \subseteq (IE1)F(X, c). \tag{2.5}$$

Proof. Firstly, $\forall x \in X$, for $(IE3)F_0(X, c)$, it is clear that

$$F_0(x, c) = f(x). \tag{2.6}$$

Secondly, $\forall x \in X$, one has $f(x) = f(\underline{c}) + f'(\zeta)(x - \underline{c})$, where $\zeta \in X$ is related to x, $f'(\zeta)(x - \underline{c}) \in B(X - \underline{c})$ follows easily from theorem 2.1, thus

$$f(x) \in (IE3)F_0(X, c). \tag{2.7}$$

So $(IE3)F_0(X, c)$ is an interval extension of $f(x)$. And also

$$\inf((IE2)F(X)) = f(c) + \inf(B(X - c)) \leq$$

$$\leq f(\underline{c}) + \sup(B(c - \underline{c})) + \inf(B(X - c)) = f(\underline{c}) + \sum_{i=1}^{n} (\sup(B_i(c_i - \underline{c}_i)) + \inf(B_i(X_i - c_i))). \tag{2.8}$$

Now, let us consider consequently in view of the different values of $\sup(B_i(c_i - \underline{c}_i)) + \inf(B_i(X_i - c_i))$:

Case 1: if $\underline{B}_i < 0 < \bar{B}_i$, then

$$\text{for } c_i > \underline{c}_i, \text{ one has } \begin{cases} \underline{B}_i(\bar{X}_i - c_i) > \bar{B}_i(\underline{X}_i - c_i), \\ \underline{B}_i(c_i - \underline{c}_i) < \bar{B}_i(c_i - \underline{c}_i); \end{cases}$$

$$\text{for } c_i < \underline{c}_i, \text{ one has } \begin{cases} \underline{B}_i(\bar{X}_i - c_i) < \bar{B}_i(\underline{X}_i - c_i), \\ \underline{B}_i(c_i - \underline{c}_i) > \bar{B}_i(c_i - \underline{c}_i); \end{cases}$$

$$\text{for } c_i = \underline{c}_i, \text{ one has } \begin{cases} \underline{B}_i(\bar{X}_i - c_i) = \bar{B}_i(\underline{X}_i - c_i), \\ \underline{B}_i(c_i - \underline{c}_i) = \bar{B}_i(c_i - \underline{c}_i) = 0. \end{cases}$$

Case 2: if $\underline{B}_i \geq 0$, then

$$\text{for } c_i \geq \underline{c}_i, \text{ one has } \begin{cases} \underline{B}_i(\bar{X}_i - c_i) \geq \bar{B}_i(\underline{X}_i - c_i), \\ \underline{B}_i(c_i - \underline{c}_i) \leq \bar{B}_i(c_i - \underline{c}_i). \end{cases}$$

Case 3: if $\bar{B}_i \leq 0$, then

$$\text{for } c_i \leq \underline{c}_i, \text{ one has } \begin{cases} \underline{B}_i(\bar{X}_i - c_i) \leq \bar{B}_i(\underline{X}_i - c_i), \\ \underline{B}_i(c_i - \underline{c}_i) \geq \bar{B}_i(c_i - \underline{c}_i). \end{cases}$$

Gathering three above cases, we deduce that

$$\sup(B_i(c_i - \underline{c}_i)) + \inf(B_i(X_i - c_i)) = \min\left\{\underline{B}_i(\bar{X}_i - \underline{c}_i), \;\; \bar{B}_i(\underline{X}_i - \underline{c}_i)\right\},$$

thus,

$$f(\underline{c})+\sum_{i=1}^{n}(\sup(B_i(c_i-\underline{c}_i))+\inf(B_i(X_i-c_i)))=$$

$$=f(\underline{c})+\sum_{i=1}^{n}\min\left\{\underline{B}_i(\bar{X}_i-\underline{c}_i),\bar{B}_i(\underline{X}_i-\underline{c}_i)\right\}=$$

$$=f(\underline{c})+\inf(B(X-\underline{c}))=\inf((IE3)F_0(X,c)). \tag{2.9}$$

With (2.8), (2.9), we can conclude that

$$\inf((IE2)F(X))\leq\inf((IE3)F_0(X,c)). \tag{2.10}$$

With the same reason, it follows that

$$\sup((IE2)F(X))\geq\sup((IE3)F_0(X,c)). \tag{2.11}$$

Then (2.7) can be deduced from(2.10), (2.11)and theorem 2.1, and the proof is completed. □

Theorem 2.3. *Suppose that a function $f:D\subseteq R^n\to R^1$ satisfies $f\in C^2(D)$. Then (1.3) the following statements over all $X\in I(D)$ are valid*

$$\inf((IE1)F(X,\underline{c}))\leq\inf((IE3)F_0(X,\underline{c})), \tag{2.12}$$

$$\sup((IE3)F_0(X,\bar{c}))\leq\sup((IE1)F(X,\bar{c})), \tag{2.13}$$

where $(IE3)F_0(X,\bar{c})$, $(IE1)F(X,\bar{c})$ are defined by (1.3) and (0.2), (0.3), respectively.

Proof. Under different cases, discuss respectively as follows

Case 1: if $\underline{F}_i'(X)\geq 0$ then $\underline{B}_i\geq 0$ that is

$$(IE3)\underline{c}_i=\underline{x}_i=(IE1)\underline{c}_i,\quad (IE3)\bar{c}_i=\bar{x}_i=(IE1)\bar{c}_i.$$

Case 2: if $\bar{F}_i'(X)\leq 0$, then $\bar{B}_i\leq 0$, that is

$$(IE3)\underline{c}_i=\bar{x}_i=(IE1)\underline{c}_i,\quad (IE3)\bar{c}_i=\underline{x}_i=(IE1)\bar{c}_i.$$

Case 3: if $0\in\text{int}(F_i'(X))$ then B_i has two cases.

a) $0\notin B_i$;

b) $0\in B_i$, at this moment, $(IE3)\bar{c}_i$ is uncertainly equal to $(IE1)\bar{c}_i$, and $(IE3)\underline{c}_i$ is uncertainly equal to $(IE2)\underline{c}_i$ too**.**

Let us denote

$$\bar{d}_i=(IE3)\bar{c}_i,\quad \underline{d}_i=(IE3)\underline{c}_i\ (i=1,2,\ldots,n),$$

$$\bar{c}_i=(IE1)\bar{c}_i,\quad \underline{c}_i=(IE1)\underline{c}_i\ (i=1,2,\ldots,n),$$

$$I=\{i|\,0\in\text{int}(F_i'(X)),\ 0\in B_i\}\quad \bar{I}=\{i|\,0\in\text{int}(F_i'(X)),\ 0\notin B_i\}.$$

For all i from $\bar{I}$ when $\underline{c}_i\geq\underline{d}_i$,

$$-f_i'(\zeta_i)(\underline{c}_i-\underline{d}_i)+\inf(B_i(X_i-\underline{d}_i))-\inf(L_i(X_i-\underline{c}_i))=$$

$$=-f_i'(\zeta_i)(\underline{c}_i-\underline{d}_i)+\bar{B}_i(\underline{x}_i-\underline{d}_i)-\bar{L}_i(\underline{x}_i-\underline{c}_i)=$$

$$=\underline{x}_i(\bar{B}_i-\bar{L}_i)+\underline{d}_i(\bar{L}_i-f_i'(\zeta_i))+\underline{d}_i(f_i'(\zeta_i)-\bar{B}_i)\geq$$

$$\geq \underline{x}_i(\bar{B}_i - \bar{L}_i) + \underline{d}_i(\bar{L}_i - f_i'(\zeta_i)) + \underline{d}_i(f_i'(\zeta_i) - \bar{B}_i) =$$
$$= \underline{x}_i(\bar{B}_i - \bar{L}_i) + \underline{d}_i(\bar{L}_i - \bar{B}_i) = (\underline{d}_i - \underline{x}_i)(\bar{L}_i - \bar{B}_i) \geq 0.$$

When $\underline{c}_i \leq \underline{d}_i$,

$$-f_i'(\zeta_i)(\underline{c}_i - \underline{d}_i) + \inf(B_i(X_i - \underline{d}_i)) - \inf(L_i(X_i - \underline{c}_i)) =$$
$$= -f_i'(\zeta_i)(\underline{c}_i - \underline{d}_i) + \underline{B}_i(\bar{x}_i - \underline{d}_i) - \underline{L}_i(\bar{x}_i - \underline{c}_i) =$$
$$= -f_i'(\zeta_i)(\underline{c}_i - \underline{d}_i) + \underline{B}_i(\bar{x}_i - \underline{d}_i) - \underline{L}_i(\bar{x}_i - \underline{c}_i) + f_i'(\zeta_i)\bar{x}_i - f_i'(\zeta_i)\bar{x}_i =$$
$$= (\bar{x}_i - \underline{d}_i)(\underline{B}_i - f_i'(\zeta_i)) + (f_i'(\zeta_i) - \underline{L}_i)(\bar{x}_i - \underline{c}_i) \geq$$
$$\geq (\bar{x}_i - \underline{d}_i)(\underline{B}_i - \underline{L}_i) \geq 0.$$

From the above discussion, it follows that

$$\inf((IE3)F_0(X,c)) - \inf((IE1)F(X)) =$$
$$= f(\underline{d}) - f(\underline{c}) + \inf\left(\sum_{i\in\bar{I}} B_i(X_i - \underline{d}_i)\right) - \inf\left(\sum_{i\in(\bar{I}\cup I)} L_i(X_i - \underline{c}_i)\right) =$$
$$= -\sum_{i\in(\bar{I}\cup I)} f_i'(\zeta_i)(\underline{c}_i - \underline{d}_i) + \inf\left(\sum_{i\in\bar{I}} B_i(X_i - \underline{d}_i)\right) - \inf\left(\sum_{i\in(\bar{I}\cup I)} L_i(X_i - \underline{c}_i)\right) \geq$$
$$\geq -\sum_{i\in\bar{I}} f_i'(\zeta_i)(\underline{c}_i - \underline{d}_i) + \inf\left(\sum_{i\in\bar{I}} B_i(X_i - \underline{d}_i)\right) - \inf\left(\sum_{i\in\bar{I}} L_i(X_i - \underline{c}_i)\right) =$$
$$= \sum_{i\in\bar{I}} (-f_i'(\zeta_i)(\underline{c}_i - \underline{d}_i) + \inf(B_i(X_i - \underline{d}_i)) - \inf(L_i(X_i - \underline{c}_i))) \geq 0,$$

that is,

$$\inf(IE1)F(X,\underline{c}) \leq \inf(IE3)F_0(X,\underline{c}).$$

With the same reason, it follows that

$$\sup(IE3)F(X,\bar{c}) \leq \sup(IE1)F(X,\bar{c}).$$

The proof is completed. □

Example. If $f(x) = x_1^2 - x_1x_2^2 + 12x_2$ $X = ([-2,-1],[-3,2])$, we estimate the range of $f(x)$ on X.

With interval extension form of $(IE3)F_0(X,c)$, we can obtain $B_1 = [-8,-6]$, $B_2 = [4,20]$, and also

$$\begin{cases} \underline{c}_1 = -1, \\ \bar{c}_1 = -2; \end{cases} \quad \begin{cases} \underline{c}_2 = -3, \\ \bar{c}_2 = 2. \end{cases}$$

By (1.3), the computation result is

$$F\,\{Wang\ Cao\}\,([-2,-1],[-3,2]) = [-26,36]$$

With the optimal centered form of Baumann, we have $F_1'(X) = [-13,4]$, $F_2'(X) = [0,20]$.

$$\begin{cases} \underline{c}_1 = -21/17, \\ \bar{c}_1 = 13/17; \end{cases} \quad \begin{cases} \underline{c}_2 = -3, \\ \bar{c}_2 = 2. \end{cases}$$

So the range of f is as follows

$$F\{Baumann\}([-2,-1],[-3,2]) = [-26.415225, 70.6470578]$$

and

$$F\{Wang\ Cao\}([-2,-1],[-3,\ 2]) \subset F\{Baumann\}([-2,-1],[-3,2])].$$

Conclusions

In this paper, we presented an optimal centered form interval extension which extended and improved the result in [1]. The optimal centered form improves the approximation of a function extension, and it can be applied in a number of problems such as bounding the range of a function $f(x)$ over a box X, solving systems of non-linear equations or non-linear inequalities, global optimization, etc. The numerical results obtained seem to indicate that the optimal centered form might lead to a considerable improvement in their numerical efficiency too.

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