ON THE NONLINEAR DIAMOND HEAT EQUATION RELATED TO THE SPECTRUM

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Рассматривается неоднородное эволюционное уравнение, содержащее первую производную по времени и оператор, образованный разностью двух бигармонических операторов разной размерности, который авторы называют Diamond-оператором. Правая часть есть липшиц-непрерывная функция решения. С помощью преобразования Фурье найдено фундаментальное решение рассматриваемого уравнения и исследованы его свойства. На этой основе дано явное решение этого уравнения в виде свертки функции правой части с фундаментальным решением, доказаны его единственность и ограниченность в равномерной норме.

Introduction

It is well known that for the heat equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \triangle u(x,t) \tag{0.1}$$

with the initial condition

$$u(x,0) = f(x)$$

where $\triangle = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator and $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, and f is a continuous function, we obtain the solution

$$u(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \int\limits_{\mathbb{R}^n} \exp\left[-\frac{|x-y|^2}{4c^2 t}\right] f(y)dy \tag{0.2}$$

as the solution of (0.1).

Now, (0.2) can be written as $u(x,t) = E(x,t) * f(x)$ where

$$E(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \exp\left[-\frac{|x|^2}{4c^2 t}\right]. \tag{0.3}$$

$E(x,t)$ is called *the heat kernel*, where $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ and $t > 0$, see [2, p. 208–209].

Moreover, we obtain $E(x,t) \to \delta$ as $t \to 0$, where δ is the Dirac-delta distribution. We also have extended (0.1) to be the equation

$$\frac{\partial}{\partial t} u(x,t) = -c^2 \triangle^2 u(x,t) \tag{0.4}$$

with the initial condition

$$u(x,0) = f(x)$$

where $\triangle^2 = \triangle\triangle$ is the biharmonic operator, that is

$$\triangle^2 = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^2 .$$

We can find the solution of (0.4) by using the n-dimensional Fourier transform to apply. We obtain

$$u(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-c^2|\xi|^4 t + i(\xi, x-y)} f(y)\, dy d\xi$$

as a solution of (0.4), or $u(x,t)$ can be written in the convolution form

$$u(x,t) = E(x,t) f(x)$$

where

$$E(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-c^2|\xi|^4 t + i(\xi,x)} d\xi \tag{0.5}$$

$|\xi|^4 = (\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2)^2$ and $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$. The function $E(x,t)$ of (0.5) is the kernel of (0.4) and also $E(x,t) \to \delta$ as $t \to 0$, since

$$\lim_{t \to 0} E(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{(\xi,x)i} d\xi = \delta(x),$$

see [3, p. 396, Eq. (10.2.19b)]. Now, the purpose of this work is to study the equation

$$\frac{\partial}{\partial t} u(x,t) - c^2 \diamondsuit u(x,t) = f(x,t,u(x,t)) \tag{0.6}$$

which is called the nonlinear diamond heat equation where $(x,t) \in \mathbb{R}^n \times (0,\infty)$ and the operator $\diamondsuit$ is first introduced by A. Kananthai [1, p. 27–37] and named *the Diamond operator* defined by

$$\diamondsuit = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} \right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2} \right)^2 , \tag{0.7}$$

$p + q = n$ is the dimension of space $\mathbb{R}^n$, $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and c is a positive constant.

We consider the equation (0.6) with the following conditions on u and f as follows.

1. $u(x,t) \in \mathcal{C}^{(4)}(\mathbb{R}^n)$ for any $t > 0$ where $\mathcal{C}^{(4)}(\mathbb{R}^n)$ is the space of continuous function with 4-derivatives.

2. f satisfies the Lipchitz condition, that is $|f(x,t,u) - f(x,t,w)| \le A|u - w|$ where A is constant with $0 < A < 1$.

3.
$$\int_0^\infty \int_{\mathbb{R}^n} |f(x,t,u(x,t))|\,dx\,dt < \infty$$

for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $0 < t < \infty$ and $u(x,t)$ is continuous function on $\mathbb{R}^n \times (0, \infty)$.

Under such conditions of f, u and for the spectrum of $E(x,t)$, we obtain the convolution

$$u(x,t) = E(x,t) f(x,t,u(x,t))$$

as a unique solution in the compact subset of $\mathbb{R}^n \times (0, \infty)$ where $E(x,t)$ is an elementary solution defined by (1.5) and is called the Diamond heat kernel.

1. Preliminaries

Definition 1.1. Let $f(x) \in \mathbb{L}_1(\mathbb{R}^n)$ — the space of integrable function in $\mathbb{R}^n$. The Fourier transform of $f(x)$ is defined by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x)\,dx \tag{1.1}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ is the usual inner product in $\mathbb{R}^n$ and $dx = dx_1\,dx_2 \ldots dx_n$.

Also, the inverse of Fourier transform is defined by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{f}(\xi) d\xi. \tag{1.2}$$

Definition 1.2. Let $E(x,t)$ be defined by (1.5) which is called the diamond heat kernel. The spectrum of $E(x,t)$ is the bounded support of the Fourier transform $\widehat{E(\xi,t)}$ for any fixed $t > 0$.

Definition 1.3. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ be a point in $\mathbb{R}^n$ and we write

$$u = \xi_1^2 + \xi_2^2 + \ldots + \xi_p^2 - \xi_{p+1}^2 - \xi_{p+2}^2 - \ldots - \xi_{p+q}^2, \quad p + q = n.$$

Denote by $\Gamma_+ = \{\xi \in \mathbb{R}^n : \xi_1 > 0 \text{ and } u > 0\}$ the set of an interior of the forward cone, and $\overline{\Gamma}_+$ denotes the closure of Γ_+.

Let Ω be spectrum of $E(x,t)$ defined by definition 1.2 for any fixed $t > 0$ and $\Omega \subset \overline{\Gamma}_+$. Let $\widehat{E(\xi,t)}$ be the Fourier transform of $E(x,t)$ and define

$$\widehat{E(\xi,t)} = \begin{cases} \dfrac{1}{(2\pi)^{n/2}} \exp\left[c^2 t \left(\left(\sum\limits_{i=1}^{p} \xi_i^2\right)^2 - \left(\sum\limits_{j=p+1}^{p+q} \xi_j^2\right)^2\right)\right] & \text{for } \xi \in \Gamma_+, \\ 0 & \text{for } \xi \notin \Gamma_+. \end{cases} \tag{1.3}$$

Lemma 1.1. *Let* L *be the operator defined by*

$$\mathrm{L} = \frac{\partial}{\partial t} - c^2 \Diamond \tag{1.4}$$

where $\diamondsuit$ is the Diamond operator defined by

$$\diamondsuit = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2}\right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2}\right)^2,$$

$p+q=n$ is the dimension of $\mathbb{R}^n$, $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $t \in (0, \infty)$ and c is a positive constant. Then we obtain

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^2\right) + i(\xi, x)\right] d\xi \qquad (1.5)$$

as an elementary solution of (1.4) which is called the diamond heat kernel in the spectrum $\Omega \subset \mathbb{R}^n$ for $t > 0$.

Proof. Let $\mathrm{L}E(x,t) = \delta(x,t)$ where $E(x,t)$ is the kernel or the elementary solution of operator L and δ is the Dirac-delta distribution. Thus

$$\frac{\partial}{\partial t} E(x,t) - c^2 \diamondsuit E(x,t) = \delta(x)\delta(t).$$

Apply the Fourier transform defined by (1.1) to the both sides of the equation, we obtain

$$\frac{\partial}{\partial t}\widehat{E(\xi,t)} - c^2 \left(\left(\sum_{i=1}^{p} \xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^2\right) \widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}}\delta(t).$$

Thus

$$\widehat{E(\xi,t)} = \frac{H(t)}{(2\pi)^{n/2}} \exp\left[c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^2\right)\right]$$

where $H(t)$ is the Heaviside function. Since $H(t) = 1$ for $t > 0$. Therefore,

$$\widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}} \exp\left[c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^2\right)\right]$$

which has been already defined by (1.3). Thus

$$E(x,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{E(\xi,t)} d\xi = \frac{1}{(2\pi)^{n/2}} \int_\Omega e^{i(\xi,x)} \widehat{E(\xi,t)} d\xi$$

where Ω is the spectrum of $E(x,t)$. Thus from (1.3)

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^2\right) + i(\xi, x)\right] d\xi$$

for $t > 0$. □

Definition 1.4. Let us extend $E(x,t)$ to $\mathbb{R}^n \times \mathbb{R}$ by setting

$$E(x,t) = \begin{cases} \dfrac{1}{(2\pi)^n} \displaystyle\int_\Omega \exp\left[c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^2\right) + i(\xi, x)\right] d\xi & \text{for } t > 0, \\ 0 & \text{for } t \leq 0. \end{cases}$$

2. Main Results

Theorem 2.1. *The kernel $E(x,t)$ defined by (1.5) have the following properties:*

1) $E(x,t) \in \mathcal{C}^\infty$ – the space of continuous function for $x \in \mathbb{R}^n$, $t > 0$ with infinitely differentiable;

2) $\left(\frac{\partial}{\partial t} - c^2\Diamond\right) E(x,t) = 0$ for $t > 0$;

3)

$$|E(x,t)| \leq \frac{2^{2-n}}{\pi^{n/2}} \frac{M(t)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)},$$

for $t > 0$ where $M(t)$ is a function of t in the spectrum Ω and Γ denote the Gamma function. Thus $E(x,t)$ is bounded for any fixed $t > 0$;

4) $\lim_{t\to 0} E(x,t) = \delta$.

Proof.

1. From (1.5), since

$$\frac{\partial^n}{\partial x^n} E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \frac{\partial^n}{\partial x^n} \exp\left[c^2 t\left(\left(\sum_{i=1}^{p} \xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^2\right) + i(\xi,x)\right] d\xi.$$

Thus $E(x,t) \in \mathcal{C}^\infty$ for $x \in \mathbb{R}^n$, $t > 0$.

2. By computing directly, we obtain

$$\left(\frac{\partial}{\partial t} - c^2\Diamond\right) E(x,t) = 0.$$

3. We have

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2 t\left(\left(\sum_{i=1}^{p} \xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^2\right) + i(\xi,x)\right] d\xi,$$

$$|E(x,t)| \leq \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2 t\left(\left(\sum_{i=1}^{p} \xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^2\right)\right] d\xi.$$

By changing to bipolar coordinates

$$\xi_1 = r\omega_1, \xi_2 = r\omega_2, \ldots, \xi_p = r\omega_p \quad \text{and} \quad \xi_{p+1} = s\omega_{p+1}, \xi_{p+2} = s\omega_{p+2}, \ldots, \xi_{p+q} = s\omega_{p+q}$$

where $\sum_{i=1}^{p} \omega_i^2 = 1$ and $\sum_{j=p+1}^{p+q} \omega_j^2 = 1$. Thus

$$|E(x,t)| \leq \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2 t\left(s^4 - r^4\right)\right] r^{p-1} s^{q-1}\, dr\, ds\, d\Omega_p\, d\Omega_q$$

where $d\xi = r^{p-1}s^{q-1}\,dr\,ds\,d\Omega_p\,d\Omega_q$, $d\Omega_p$ and $d\Omega_q$ are the elements of surface area of the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. Since $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ and we suppose $0 \le r \le R$ and $0 \le s \le L$ where R and L are constants. Thus we obtain

$$|E(x,t)| \le \frac{\Omega_p\,\Omega_q}{(2\pi)^n} \int_0^R \int_0^L \exp\left[c^2 t\left(s^4 - r^4\right)\right] r^{p-1}s^{q-1}\,ds\,dr = \frac{\Omega_p\,\Omega_q}{(2\pi)^n} M(t)$$

for any fixed $t > 0$ in the spectrum

$$\Omega = \frac{2^{2-n}}{\pi^{n/2}} \frac{M(t)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)} \tag{2.1}$$

where

$$M(t) = \int_0^R \int_0^L \exp\left[c^2 t\left(s^4 - r^4\right)\right] r^{p-1}s^{q-1}\,ds\,dr \tag{2.2}$$

is a function of $t > 0$, $\Omega_p = \dfrac{2\pi^{p/2}}{\Gamma\left(\frac{p}{2}\right)}$ and $\Omega_q = \dfrac{2\pi^{q/2}}{\Gamma\left(\frac{q}{2}\right)}$. Thus, for any fixed $t > 0$, $E(x,t)$ is bounded.

4. By (1.5), we have

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2 t\left(\left(\sum_{i=1}^{p} \xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^2\right) + i(\xi,x)\right] d\xi.$$

Since $E(x,t)$ exists, then

$$\lim_{t\to 0} E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega e^{i(\xi,x)} d\xi = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(\xi,x)} d\xi = \delta(x), \quad \text{for } x \in \mathbb{R}^n.$$

See [3, p. 396, Eq. (10.2.19b)]. □

Theorem 2.2. *Given the nonlinear equation*

$$\frac{\partial}{\partial t} u(x,t) - c^2 \Diamond u(x,t) = f(x,t,u(x,t)) \tag{2.3}$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$ and with the following conditions on u and f as follows:

1) $u(x,t) \in \mathcal{C}^{(4)}(\mathbb{R}^n)$ for any $t > 0$ where $\mathcal{C}^{(4)}(\mathbb{R}^n)$ is the space of continuous function with 4-derivatives;

2) f satisfies the Lipchitz condition, that is $|f(x,t,u) - f(x,t,w)| \le A|u - w|$ where A is constant and $0 < A < 1$;

3)

$$\int_0^\infty \int_{\mathbb{R}^n} |f(x,t,u(x,t))|\,dx\,dt < \infty$$

for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $t \in (0,\infty)$ and $u(x,t)$ is continuous function on $\mathbb{R}^n \times (0,\infty)$.

Then we obtain the convolution

$$u(x,t) = E(x,t)f(x,t,u(x,t)) \tag{2.4}$$

as a unique solution of (2.3) for $x \in \Omega_0$ *where* Ω_0 *is an compact subset of* $\mathbb{R}^n$, $0 \leq t \leq T$ *with* T *is constant and* $E(x,t)$ *is an elementary solution defined by (1.5) and also* $u(x,t)$ *is bounded.*

In particular, if we put $p = 0$ *in (2.3) then (2.3) reduces to the nonlinear equation*

$$\frac{\partial}{\partial t} u(x,t) - c^2 \triangle^2 u(x,t) = f(x,t,u(x,t))$$

which is related to the heat equation.

Proof. Convolving both sides of (2.3) with $E(x,t)$ and then we obtain the solution

$$u(x,t) = E(x,t)f(x,t,u(x,t))$$

or

$$u(x,t) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} E(r,s) f(x-r, t-s, u(x-r,t-s))\, dr\, ds$$

where $E(r,s)$ is given by Definition 1.4.

We next show that $u(x,t)$ is bounded on $\mathbb{R}^n \times (0,\infty)$. We have

$$|u(x,t)| \leq \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |E(r,s)|\, |f(x-r,t-s,u(x-r,t-s))|\, dr\, ds \leq \frac{2^{2-n}}{\pi^{n/2}} \frac{NM(t)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}$$

by the condition 3 and (2.1) where

$$N = \int_{0}^{\infty} \int_{\mathbb{R}^n} |f(x,t,u(x,t))|\, dx\, dt.$$

Thus $u(x,t)$ is bounded on $\mathbb{R}^n \times (0,\infty)$.

To show that $u(x,t)$ is unique, suppose there is another solution $w(x,t)$ of equation (2.3). Let the operator

$$\mathrm{L} = \frac{\partial}{\partial t} - c^2 \diamondsuit$$

then (2.3) can be written in the form

$$\mathrm{L}\, u(x,t) = f(x,t,u(x,t)).$$

Thus

$$\mathrm{L}\, u(x,t) - \mathrm{L}\, w(x,t) = f(x,t,u(x,t)) - f(x,t,w(x,t)).$$

By the condition 2 of the Theorem,

$$|\mathrm{L}\, u(x,t) - \mathrm{L}\, w(x,t)| \leq A|u(x,t) - w(x,t)|. \tag{2.5}$$

Let $\Omega_0 \times (0,T]$ be compact subset of $\mathbb{R}^n \times (0,\infty)$ and $\mathrm{L} : \mathcal{C}^{(4)}(\Omega_0) \longrightarrow \mathcal{C}^{(4)}(\Omega_0)$ for $0 \leq t \leq T$.

Now $\left(\mathcal{C}^{(4)}(\Omega_0), \|\cdot\|\right)$ is a Banach space where $u(x,t) \in \mathcal{C}^{(4)}(\Omega_0)$ for $0 \le t \le T$, $\|\cdot\|$ given by

$$\|u(x,t)\| = \sup_{x \in \Omega_0} |u(x,t)|.$$

Then, from (2.5) with $0 < A < 1$, the operator L is a contraction mapping on $\mathcal{C}^{(4)}(\Omega_0)$. Since $\left(\mathcal{C}^{(4)}(\Omega_0), \|\cdot\|\right)$ is a Banach space and $\mathrm{L} : \mathcal{C}^{(4)}(\Omega_0) \to \mathcal{C}^{(4)}(\Omega_0)$ is a contraction mapping on $\mathcal{C}^{(4)}(\Omega_0)$, by Contraction Theorem, see [4, p. 300], we obtain the operator L has a fixed point and has uniqueness property. Thus $u(x,t) = w(x,t)$. It follows that the solution $u(x,t)$ of (2.3) is unique for $(x,t) \in \Omega_0 \times (0,T]$ where $u(x,t)$ is defined by (2.4).

In particular, if we put $p = 0$ in (2.3) then (2.3) reduces to the nonlinear equation

$$\frac{\partial}{\partial t} u(x,t) - c^2 \triangle^2 u(x,t) = f(x,t,u(x,t))$$

which has solution

$$u(x,t) = E(x,t) f(x,t,u(x,t))$$

where $E(x,t)$ is defined by (1.5) with $p = 0$. That is complete of proof. □

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